Identical and Nonidentical Synchronization of Hyperchaotic Systems by Active Backstepping Method

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Abstract: This paper deals with the tracking and synchronization problems of hyperchaotic systems based on active backstepping method. The mentioned method consists of a recursive approach that interlaces the choice of a Lyapunov function with the design of feedback control. At first, a nonlinear recursive active backstepping control vector is designed such that state variables of the hyperchaotic Wang system track desired trajectories. Furthermore, this method is applied to synchronize two identical hyperchaotic Wang systems. Eventually, it is used to implement synchronization between hyperchaotic Wang system and hyperchaotic Rössler system. Numerical simulations are performed to demonstrate the effectiveness and efficiency of the three designed active backstepping control vectors.

Keywords: Hyperchaotic System, Identical Synchronization, Backstepping Methods, Rössler and Wang Systems.

1 Introduction

In general, a hyperchaotic system is defined as a chaotic system with at least two positive exponents, implying that its dynamics are extended in several different directions simultaneously [1]. The first hyperchaotic system was introduced by Rössler [2] which represents dynamical equations of a particular chemical reaction. Recently, many new hyperchaotic systems were introduced by scientists of various fields [3-7] and synchronization problem of the hyperchaotic systems has become a hot topic among researchers [8-14] due to wide applications of synchronization of hyperchaotic systems in many engineering field such as secure communications [15-17], image encryption [18, 19], lasers [20, 21].

In recent decades, different methods have been proposed to synchronize hyperchaotic systems including adaptive control [22], fuzzy control [23], active control [14] sliding mode control [24], impulsive control [8], backstepping method [25-30] and so on.

In particular, backstepping recursive nonlinear control scheme has been widely employed to synchronize and control of hyperchaotic systems which can be continuous-time [25, 26] or discrete-time [27]. It is well known that backstepping method represents a powerful and systematic technique that recursively interlaces the choice of a Lyapunov function with the design of feedback control. The mentioned method can guarantee global stability, tracking and transient performance for a broad class of strict-feedback nonlinear systems and control goal can be achieved with reduced control effort. Motivated above discussion, in this paper three novel active control vectors based on the backstepping method are proposed to control and synchronize the Wang and Rössler hyperchaotic systems. The advantages of proposed method can be summarized as follows: (a) it is a systematic procedure for hyperchaos control and synchronization of hyperchaotic systems and guarantees the stability of the closed-loop system; (b) The proposed method can overcome the controller singularity problem resulting from the nonlinear term of quadratic type; (c) it can be applied to a variety of hyperchaotic systems no matter whether it contains external excitation or not; (d) there is no derivatives in controllers, so it is easy to be implemented.

The rest of this paper is organized as follows. Section 2 presents two dynamic models of the hyperchaotic Wang and Rössler systems. In Section 3, the tracking problem for hyperchaotic Wang system is investigated by using backstepping method. In Section 4, an active backstepping control vector is designed to synchronize two identical hyperchaotic Wang systems. In Section 5, the generalized synchronization between the hyperchaotic Rössler system and the hyperchaotic Wang system is achieved by using the backstepping

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controllers. Numerical simulation results which that confirm the validity and feasibility of the designed controllers are shown in Section 6. Finally, conclusions are given in Section 7.

2 Descriptions of Wang and Rössler Systems

In order to synchronize hyperchaotic Wang and Rössler systems, a brief review of their models is expressed in this section. In 1979, Rössler [2] introduced a four-dimensional hyperchaotic system containing only one nonlinear quadratic term. Due to its simplicity, the Rössler system has become a standard benchmark hyperchaotic system to verify different control methods of hyperchaos phenomenon. Numerical calculations show that the mentioned system has two positive Lyapunov exponents, namely, $L_1 = 0.11$ and $L_1 = 0.02$. In fact, the Rössler system describes a chemical reaction scheme [2]. The differential equations of Rössler system are given by

$$\begin{aligned} \dot{y}_1 &= \beta y_2 + \gamma y_1 \\ \dot{y}_2 &= 3 + y_2 y_3 \\ \dot{y}_3 &= -y_2 - y_4 \\ \dot{y}_4 &= y_1 + y_3 + \alpha y_4 \end{aligned} \tag{1}$$

where $y_i, i = 1,\ldots,4$ are the state variables of Rössler system, and α, β, γ are constant parameters. When parameters are considered as $\alpha = 0.25$, $\beta = -0.5$ and $\gamma = 0.05$, system (1) shows hyperchaotic behaviors. Figs. 1(a), (b), (c) depict the projections of hyperchaotic attractor of the system (1) onto $x_2 - x_3$, $x_2 - x_4$, $x_3 - x_4$ planes, respectively. Fig. 1(d) displays the trajectory of the system (1) plotted in $x_2 - x_3 - x_4$ space.

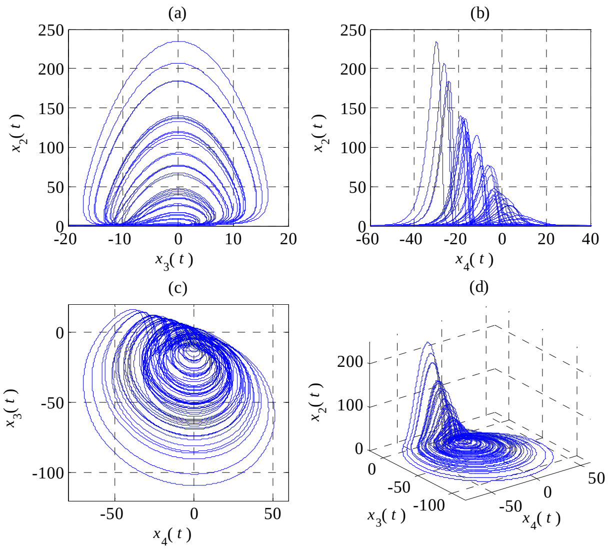

Fig. 1 Projections of hyperchaotic attractor of the Rössler system: (a) Projection onto $x_2 - x_3$ plane, (b) Projection onto $x_2 - x_4$ plane, (c) Projection of onto $x_3 - x_4$ plane, (d) Three-dimensional view in $x_2 - x_3 - x_4$ space.

Wang [31] proposed a new hyperchaotic system by introducing an additional state into the third-order Liu chaotic system [32]. The four-dimensional autonomous hyperchaotic system is described by

$$\begin{aligned} \dot{x}_1 &= -cx_1 + hx_2^2 \\ \dot{x}_2 &= a(x_3 - x_2) \\ \dot{x}_3 &= bx_2 - kx_1x_2 + x_4 \\ \dot{x}_4 &= -dx_2 \end{aligned} \tag{2}$$

where $x_i, i = 1,\ldots,4$ are the state variables of Rössler system, and a,b,c,d,h,k are positive constant parameters. When parameters are $a = 10$, $b = 40$, $c = 2.5$, $d = 10.6$, $k = 1$ and $h = 4$, system (2) is hyperchaotic with two positive Lyapunov exponent $L_1 = 1.15$, $L_2 = 0.13$. The system (2) has only one equilibrium point at the origin, and the equilibrium point is an unstable saddle node considering mentioned parameters [33]. Figs. 2(a), (b), (c) show the projections of hyperchaotic attractor of Wang system onto $x_1 - x_2$, $x_2 - x_3$ and $x_3 - x_4$ planes respectively. Fig. 2(d) depicts the trajectory of Wang system plotted in $x_2 - x_3 - x_4$ space.

3 Tracking Problem for Wang System

In this section, backstepping method is used to design active controllers to suppress hyperchaos in the hyperchaotic Wang system. The controlled hyperchaotic Wang system [31] is given by

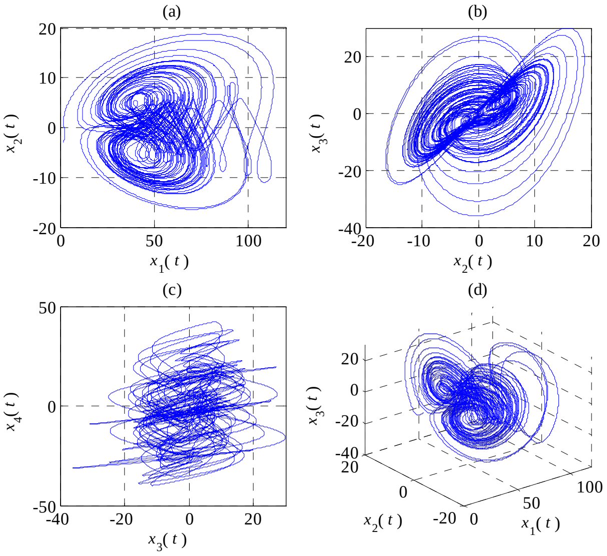

Fig. 2 Projections of hyperchaotic attractor of the system (2): (a) Projection onto $x_1 - x_2$ plane, (b) Projection onto $x_2 - x_3$ plane, (c) Projection onto $x_3 - x_4$ plane, (d) Three-dimensional view of hyperchaotic attractor.

$$\begin{aligned}\dot{x}_1 &= -cx_1 + hx_2^2 + u_1\\ \dot{x}_2 &= a(x_3 - x_2) + u_2\\ \dot{x}_3 &= bx_2 - kx_1x_2 + x_4 + u_3\\ \dot{x}_4 &= -dx_2 + u_4\end{aligned} \tag{3}$$

Our main goal is to design an appropriate active backstepping control vector $\boldsymbol{u} = [u_1, u_2, u_3, u_4]^T$ such that the controlled system (3) can track desired trajectory $x_d(t)$ with a scalar output $x_1(t)$ in the sense that

$$\lim_{t\to\infty}\left|x_1 - x_d\right| = 0 \tag{4}$$

The backstepping design procedure is recursive. At the i^{th} step, the i^{th}-order subsystem is stabilized with respect to a Lyapunov function by the design of a virtual control φ_i and a control input u_i .Now, we begin to design the active controllers based on the backstepping design method as follows.

Step 1. Assume that $\eta_1 = x_1 - x_d$, then its derivative is derived as

$$\dot{\eta}_1 = \dot{x}_1 - \dot{x}_d = -c(\eta_1 + x_d) + hx_2^2 - \dot{x}_d + u_1 \tag{5}$$

where $x_2 = \varphi_1(\eta_1)$ is considered as a virtual controller. To design $\varphi_1(\eta_1)$ such that stabilizes η_1-subsystem (5), the following Lyapunov function V_1 is selected.

$$V_1 = 0.5\eta_1^2 \tag{6}$$

Then, the derivative of V_1 is concluded as

$$\dot{V}_1 = \eta_1\dot{\eta}_1 = \eta_1(-c\eta_1 + h\varphi_1^2(\eta_1) - cx_d - \dot{x}_d + u_1) \tag{7}$$

Here, we choose $\varphi_1(\eta_1) = 0$ and $u_1 = cx_d + \dot{x}_d$ such that $\dot{V}_1 = -c\eta_1^2$ is negative definite. This implies that the η_1-subsystem (5) is stable. Since the virtual control function $\varphi_1(\eta_1)$ is estimative, an error variable is defined as follows

$$\eta_2 = x_2 - \varphi_1(\eta_1) \tag{8}$$

The following $(\eta_1 - \eta_2)$-subsystem can be obtained as

$$\begin{aligned}\dot{\eta}_1 &= -c\eta_1 + h\eta_2^2\\ \dot{\eta}_2 &= a\,x_3 - a\,\eta_2 + u_2\end{aligned} \tag{9}$$

where $x_3 = \varphi_2(\eta_1, \eta_2)$ is regarded as a virtual controller.

Step 2. In order to stabilize the $(\eta_1 - \eta_2)$-subsystem (9), the Lyapunov function V_2 is defined as follows.

$$V_2 = V_1 + 0.5\eta_2^2 \tag{10}$$

The derivative of V_2 is yielded as

$$\dot{V}_2 = -c\eta_1^2 + h\eta_1\eta_2^2 + a\eta_2\varphi_2(\eta_1, \eta_2) - a\eta_2^2 + \eta_2 u_2 \tag{11}$$

If $\varphi_2(\eta_1, \eta_2) = 0$ and $u_2 = -h\,\eta_1\,\eta_2$, then $\dot{V}_2 = -c\eta_1^2 - a\eta_2^2 < 0$ concludes that the $(\eta_1 - \eta_2)$-subsystem (9) is stable. Since the virtual control function $\varphi_2(\eta_1, \eta_2)$ is estimative, an error variable is defined as follows.

$$\eta_3 = x_3 - \varphi_2(\eta_1, \eta_2) \tag{12}$$

Then the following $(\eta_1 - \eta_2 - \eta_3)$-subsystem is obtained as

$$\begin{aligned}\dot{\eta}_1 &= -c\,\eta_1 + h\,\eta_2^2\\ \dot{\eta}_2 &= a\,(\eta_3 - \eta_2) - h\,\eta_1\,\eta_2\\ \dot{\eta}_3 &= b\,\eta_2 - k\,\eta_2(\eta_1 + x_d) + x_4 + u_3\end{aligned} \tag{13}$$

where $x_4 = \varphi_3(\eta_1, \eta_2, \eta_3)$ is regarded as a virtual controllers.

Step 3. In order to stabilize the $(\eta_1 - \eta_2 - \eta_3)$-subsystem (13), the Lyapunov function V_3 is selected as follows.

$$V_3 = V_1 + V_2 + 0.5\eta_3^2 \tag{14}$$

The derivative of V_3 is given by

$$\begin{aligned}\dot{V}_3 = &-c\eta_1^2 - a\eta_2^2 + (a+b)\eta_2\eta_3 - k\eta_1\eta_2\eta_3\\ &- k\eta_2\eta_3 x_d + \eta_3\varphi_3(\eta_1, \eta_2, \eta_3) + \eta_3 u_3\end{aligned} \tag{15}$$

Then, we choose $\varphi_3(\eta_1, \eta_2, \eta_3)$ and $u_3 = -\left(a+b\right)\eta_2 + k\eta_1\eta_2 + k\eta_2 x_d - \eta_3$ such that $\dot{V}_3 = -c\eta_1^2 - a\eta_2^2 - \eta_3^2 < 0$ is negative definite. This implies that the $(\eta_1 - \eta_2 - \eta_3)$-subsystem (13) is stable. Similarly, assume that $\eta_4 = x_4 - \varphi_3\left(\eta_1, \eta_2, \eta_3\right)$ then the following subsystem is concluded.

$$\begin{aligned}\dot{\eta}_1 &= -c\eta_1 + h\eta_2^2\\ \dot{\eta}_2 &= a\,(\eta_3 - \eta_2) - h\eta_1\eta_2\\ \dot{\eta}_3 &= -a\eta_2 + \eta_4 - \eta_3\\ \dot{\eta}_4 &= -d\eta_2 + u_4\end{aligned} \tag{16}$$

Step 4. In order to stabilize the $(\eta_1 - \eta_2 - \eta_3 - \eta_4)$-system (16), a Lyapunov function V_4 is considered as follows.

$$V_4 = V_1 + V_2 + V_3 + 0.5\,\eta_4^2 \tag{17}$$

The derivative of V_4 is given by

$$\dot{V}_4 = -c\,\eta_1^2 - a\,\eta_2^2 - \eta_3^2 + \eta_3\,\eta_4 - d\,\eta_2\,\eta_4 + \eta_4\,u_4 \tag{18}$$

If we choose $u_4 = -\eta_4 - \eta_3 + d\eta_2$, then $\dot{V}_4 = -c\eta_1^2 - a\eta_2^2 - \eta_3^2 - \eta_4^2 < 0$, therefore $(\eta_1 - \eta_2 - \eta_3 - \eta_4)$-system (16) is stable. Since $\dot{V}_4$ is negative definite, it follows that in the

$(\eta_1-\eta_2-\eta_3-\eta_4)$ coordinates the equilibrium $(0,0,0,0)$ is stable. By considering $\eta_1=x_1-x_d(t)$, $\eta_2=x_2$, $\eta_3=x_3$ and $\eta_4=x_4$, one can clearly conclude that the state variable $x_1(t)$ of controlled Wang system (3), tracks the desired trajectory $x_d(t)$.

4 Synchronization of Two Identical Wang Systems

In this section, two identical hyperchaotic Wang systems are considered as master and slave systems respectively. The master system is described as Eq. (2), and the slave system is given by

$$\begin{aligned}\dot y_1&=-cy_1+hy_2^2+u_1\\ \dot y_2&=a(y_3-y_2)+u_2\\ \dot y_3&=by_2-ky_1y_2+y_4+u_3\\ \dot y_4&=-dy_2+u_4\end{aligned}\tag{19}$$

where $\boldsymbol{x}=[x_1,x_2,x_3,x_4]^T$ stands for the state variables vector of the master system, $\boldsymbol{y}=[y_1,y_2,y_3,y_4]^T$ denotes the state variables vector of the slave system and $\boldsymbol{u}=[u_1,u_2,u_3,u_4]^T$ is the control inputs vector.

Our goal is to design an appropriate active backstepping control inputs vector $\boldsymbol{u}(t)\in\mathbb{R}^4$ such that the state variables of the slave system track the ones of the master system in the sense that

$$\lim_{t\to\infty}\|\boldsymbol{y}-\boldsymbol{x}\|=0\tag{20}$$

where $\|.\|$ is the Euclidean norm. By defining the synchronization errors vector as $\boldsymbol{e}=[e_1,e_2,e_3,e_4]^T, e_i=y_i-x_i$, i = 1, 2, 3, 4, we can subtract Eq. (19) from Eq. (2), which yields the following synchronization error system.

$$\begin{aligned}\dot e_1&=-ce_1+he_2^2+2hx_2e_2+u_1\\ \dot e_2&=a(e_3-e_2)+u_2\\ \dot e_3&=be_2+e_4-k(e_1e_2+x_1e_2+x_2e_1)+u_3\\ \dot e_4&=-de_2+u_4\end{aligned}\tag{21}$$

Therefore, the identical synchronization problem is reduced to the stabilization problem of the synchronization error system (21). Consequently, the control inputs vector $\boldsymbol{u}(t)$ should be designed to stabilize the system (21). Finally, the stabilization of the system (21) implies that the master and slave Wang systems are synchronized properly. Now, the active controllers will be designed based on the backstepping method as follows.

Step1. Let $\mu_1=e_1$, then its derivative can be obtained a

$$\dot\mu_1=-c\mu_1+he_2^2+2hx_2e_2+u_1\tag{22}$$

where $e_2=\varphi_1(\mu_1)$ can be regarded as a virtual controller. To design of $\varphi_1(\mu_1)$ and u_1 such that stabilize μ_1-subsystem (22), the following Lyapunov function V_1 is considered as follows.

$$V_1=0.5\mu_1^2\tag{23}$$

Then, the derivative of V_1 is obtained as

$$\dot V_1=\mu_1[h\varphi_1^2(\mu_1)+2hx_2\varphi_1(\mu_1)-c\mu_1+u_1]\tag{24}$$

If we choose $\varphi_1(\mu_1)=u_1=0$, then $\dot V_1=-c\mu_1^2<0$ is negative definite. This implies that the μ_1-subsystem (22) is stable. Since the virtual control function $\varphi_1(\mu_1)$ is estimative, the error between e_2 and $\varphi_1(\mu_1)$ is defined as

$$\mu_2=e_2-\varphi_1(\mu_1)\tag{25}$$

Then, the $(\mu_1-\mu_2)$-subsystem can be derived as

$$\begin{aligned}\dot\mu_1&=-c\mu_1+h\mu_2^2+2hx_2\mu_2\\ \dot\mu_2&=ae_3-a\mu_2+u_2\end{aligned}\tag{26}$$

where $e_3=\varphi_2(\mu_1,\mu_2)$ can be considered as a virtual controller.

Step 2. In order to stabilize the $(\mu_1-\mu_2)$-subsystem (26), the Lyapunov function V_2 is defined as

$$V_2=V_1+0.5\mu_2^2\tag{27}$$

The derivative of V_2 is given by

$$\begin{aligned}\dot V_2&=h\mu_1\mu_2^2+2hx_2\mu_1\mu_2-c\mu_1^2-a\mu_2^2+\\ &\quad+a\mu_2\varphi_2(\mu_1,\mu_2)+\mu_2u_2\end{aligned}\tag{28}$$

If $\varphi_2(\mu_1,\mu_2)=0$ and $u_2=-h\mu_1\mu_2-2hx_2\mu_1$, then $\dot V_2=-c\mu_1^2-a\mu_2^2<0$, which implies the $(\mu_1-\mu_2)$-subsystem (26) is stable. Similarly, assume that $\mu_3=e_3-\varphi_2(\mu_1,\mu_2)$, then the following $(\mu_1-\mu_2-\mu_3)$-system can be derived.

$$\begin{aligned}\dot\mu_1&=-c\mu_1+h\mu_2^2+2hx_2\mu_2\\ \dot\mu_2&=a\mu_3-a\mu_2-h\mu_1\mu_2-2hx_2\mu_1\\ \dot\mu_3&=b\mu_2+e_4-k(\mu_1\mu_2+x_1\mu_2+x_2\mu_1)+u_3\end{aligned}\tag{29}$$

where $e_4=\varphi_3(\mu_1,\mu_2,\mu_3)$ can be regarded as a virtual controller.

Step 3. The Lyapunov function V_3 is defined by

$$V_3=V_2+V_1+0.5\mu_3^2\tag{30}$$

In order to make $(\mu_1-\mu_2-\mu_3)$ subsystem (29) stable, the derivative of V_3 is given by

$$\begin{aligned}\dot{V}_3 &= -c\mu_1^2 - a\mu_2^2 + (a+b)\mu_2\mu_3 - kx_1\mu_2\mu_3 \\ &\quad - kx_2\mu_1\mu_3 + \mu_3\varphi_3(\mu_1,\mu_2,\mu_3) + \mu_3 u_3\end{aligned} \tag{31}$$

We can choose $\varphi_3(\mu_1,\mu_2,\mu_3) = -\mu_3$ and $u_3 = -(a+b)\mu_2 + k\mu_1\mu_2 + kx_1\mu_2 + kx_2\mu_1$, so that $\dot{V}_3 = -c\mu_1^2 - a\mu_2^2 - \mu_3^2 < 0$, which verifies the $(\mu_1 - \mu_2 - \mu_3)$ -subsystem (29) is stable.

Let $\mu_4 = e_4 - \varphi_3(\mu_1,\mu_2,\mu_3)$, then the following $(\mu_1 - \mu_2 - \mu_3 - \mu_4)$ -system is concluded.

$$\begin{aligned}\dot{\mu}_1 &= -c\mu_1 + h\mu_2^2 + 2hx_2\mu_2 \\ \dot{\mu}_2 &= a\,\mu_3 - a\mu_2 - h\mu_1\mu_2 - 2hx_2\mu_1 \\ \dot{\mu}_3 &= -a\mu_2 + \mu_4 - \mu_3 \\ \dot{\mu}_4 &= -(d+a)\mu_2 + \mu_4 - \mu_3 + u_4\end{aligned} \tag{32}$$

Step 4. In order to stabilize the $(\mu_1 - \mu_2 - \mu_3 - \mu_4)$ -system (32), the Lyapunov function V_4 defined by Eq. (33).

$$V_4 = V_1 + V_2 + V_3 + 0.5\,\mu_4^2 \tag{33}$$

The derivative of V_4 is expressed as

$$\begin{aligned}\dot{V}_4 &= -c\mu_1^2 - a\mu_2^2 - \mu_3^2 + \mu_4^2 \\ &\quad - (d+a)\mu_2\mu_4 + \mu_4 u_4\end{aligned} \tag{34}$$

If $u_4 = (d+a)\mu_2 - 2\mu_4$, then $\dot{V}_4 = -c\mu_1^2 - a\mu_2^2 - \mu_3^2 - \mu_4^2 < 0$ proves that the $(\mu_1 - \mu_2 - \mu_3 - \mu_4)$ -system (32) is stable. Finally, the $(\mu_1 - \mu_2 - \mu_3 - \mu_4)$ -system is described by

$$\begin{aligned}\dot{\mu}_1 &= -c\mu_1 + h\mu_2^2 + 2h\,x_2\,\mu_2 \\ \dot{\mu}_2 &= a\,\mu_3 - a\mu_2 - h\mu_1\mu_2 - 2h\,x_2\,\mu_1 \\ \dot{\mu}_3 &= -a\mu_2 + \mu_4 - \mu_3 \\ \dot{\mu}_4 &= -\mu_4 - \mu_3\end{aligned} \tag{35}$$

Since $\dot{V}_4$ is negative definite, it follows that in the $(\mu_1 - \mu_2 - \mu_3 - \mu_4)$ coordinates the equilibrium $(0,0,0,0)$ is stable. Consequently, by considering the $\mu_1 = e_1$, $\mu_2 = e_2$, $\mu_3 = e_3 - \varphi_2(\mu_1,\mu_2) = e_3$ and $\mu_4 = e_4 - \varphi_3(\mu_1,\mu_2,\mu_3) = e_4 + e_3$, one can obtain that synchronization errors e_1, e_2, e_3, e_4 converge to zero. In other words, the state variables of the slave system (19) track the ones of the master system (2).

5 Synchronization of Two Wang and Rössler Systems

In this section, the backstepping method is employed to synchronize the hyperchaotic Rössler system with the hyperchaotic Wang system by designing an active control inputs vector . The Wang system is regarded as the master system which expressed as Eq. (2), and the hyperchaotic Rössler system is considered as the slave system which is given by

$$\begin{aligned}\dot{y}_1 &= \beta\,y_2 + \gamma\,y_1 + u_1 \\ \dot{y}_2 &= 3 + y_2 y_3 + u_2 \\ \dot{y}_3 &= -y_2 - y_4 + u_3 \\ \dot{y}_4 &= y_1 + y_3 + \alpha\,y_4 + u_4\end{aligned} \tag{36}$$

where $y_i (i = 1,2,3,4)$ are the state variables of the Rössler system and $\boldsymbol{u} = [u_1, u_2, u_3, u_4]^T$ is the control inputs vector. Similarly, by defining the synchronization errors vector as $\boldsymbol{e} = [e_1, e_2, e_3, e_4]^T, e_i = y_i - x_i$, we can subtract Eq. (36) from Eq. (2), which yields the synchronization error system.

$$\begin{aligned}\dot{e}_1 &= \beta e_2 + \gamma e_1 + f_1 + u_1 \\ \dot{e}_2 &= 3 + e_2 e_3 + x_2 e_3 + x_3 e_2 + f_2 + u_2 \\ \dot{e}_3 &= -e_2 - e_4 + f_3 + u_3 \\ \dot{e}_4 &= e_1 + e_3 + \alpha\,e_4 + f_4 + u_4\end{aligned} \tag{37}$$

where $f_1 = \beta x_2 + (\gamma + c)x_1 - hx_2^{\,2}$, $f_2 = a(x_2 - x_3) + x_2 x_3$, $f_3 = -(b+1)x_2 - 2x_4 + kx_1 x_2$ and $f_4 = dx_2 + x_1 + x_3 + \alpha x_4$. The active backstepping control inputs vector will be designed as follows.

Step 1. Assume that $\sigma_1 = e_1$, then its derivative can be obtained as

$$\dot{\sigma}_1 = \beta e_2 + \gamma\sigma_1 + f_1 + u_1 \tag{38}$$

where $e_2 = \varphi_1(\sigma_1)$ is regarded as a virtual controller. To design of $\varphi_1(\sigma_1)$ and u_1 such that stabilize the σ_1-subsystem (38), the Lyapunov function V_1 is defined as

$$V_1 = 0.5\sigma_1^2 \tag{39}$$

The derivative of V_1 is obtained as

$$\dot{V}_1 = \sigma_1[\beta\,\varphi_1(\sigma_1) + \gamma\sigma_1 + f_1 + u_1] \tag{40}$$

By considering $\varphi_1(\sigma_1) = 0$ and $u_1 = -f_1 - (\gamma + 1)\sigma_1$, the $\dot{V}_1 = -\sigma_1^2 < 0$ is negative definite. This implies that the σ_1-subsystem (38) is stable. Since the virtual control function $\varphi_1(\sigma_1)$ is estimative, the error between e_2 and $\varphi_1(\sigma_1)$ is defined as follows.

$$\sigma_2 = e_2 - \varphi_1(\sigma_1) \tag{41}$$

Thus, the σ_1-subsystem can be obtained as

$$\begin{aligned}\dot{\sigma}_1 &= \beta\sigma_2 - \sigma_1 \\ \dot{\sigma}_2 &= 3 + \sigma_2 e_3 + x_2 e_3 + x_3\sigma_2 + f_2 + u_2\end{aligned} \tag{42}$$

where $e_3 = \varphi_2(\sigma_1, \sigma_2)$ is regarded as a virtual controller.

Step 2. In order to stabilize the $(\sigma_1-\sigma_2)$-subsystem (42), the Lyapunov function V_2 is defined by

$$V_2 = V_1 + 0.5\sigma_2^2 \tag{43}$$

The derivative of V_2 is given by

$$\begin{aligned}\dot{V}_2 &= -\sigma_1^2 + \sigma_2(\sigma_2 + x_2)\varphi_2(\sigma_1,\sigma_2) + \beta\sigma_1\sigma_2 \\ &\quad + 3\sigma_2 + x_3\sigma_2^2 + \sigma_2 f_2 + \sigma_2 u_2\end{aligned} \tag{44}$$

If $\varphi_2(\sigma_1,\sigma_2)=0$ and $u_2 = -f_2 - 3 - \sigma_2 x_3 - \beta\sigma_1 - \sigma_2$, then $\dot{V}_2 = -\sigma_1^2 - \sigma_2^2 < 0$, which implies that $(\sigma_1-\sigma_2)$-subsystem (42) is stable. Similarly, assume that $\sigma_3 = e_3 - \varphi_2(\sigma_1,\sigma_2)$, then the $(\sigma_1-\sigma_2-\sigma_3)$-subsystem is concluded as

$$\begin{aligned}\dot{\sigma}_1 &= \beta\sigma_2 - \sigma_1 \\ \dot{\sigma}_2 &= (\sigma_2 + x_2)\sigma_3 - \sigma_2 - \beta\sigma_1 \\ \dot{\sigma}_3 &= -\sigma_2 - e_4 + f_3 + u_3\end{aligned} \tag{45}$$

where $e_4 = \varphi_3(\sigma_1,\sigma_2,\sigma_3)$ is regarded as a virtual controller.

Step 3. In order to stabilize the $(\sigma_1-\sigma_2-\sigma_3)$-subsystem (45), the Lyapunov function V_3 is selected as

$$V_3 = V_2 + V_1 + 0.5\sigma_3^2 \tag{46}$$

The derivative of V_3 is obtained as

$$\begin{aligned}\dot{V}_3 &= -\sigma_1^2 - \sigma_2^2 - \sigma_3\varphi_3(\sigma_1,\sigma_2,\sigma_3) \\ &\quad + \sigma_3(\sigma_2^2 + \sigma_2 x_2 - \sigma_2) + \sigma_3 f_3 + \sigma_3 u_3\end{aligned} \tag{47}$$

By selecting $\varphi_3(\sigma_1,\sigma_2,\sigma_3) = 0$ and $u_3 = -f_3 - \sigma_2{}^2 - \sigma_2 x_2 + \sigma_2 - \sigma_3$, then the $\dot{V}_3 = -\sigma_1^2 - \sigma_2^2 - \sigma_3^2 < 0$ is negative definite. This implies that the $(\sigma_1-\sigma_2-\sigma_3)$-subsystem (45) is stable. Similarly, assume that $\sigma_4 = e_4 - \varphi_3(\sigma_1,\sigma_2,\sigma_3)$, then the $(\sigma_1-\sigma_2-\sigma_3-\sigma_4)$-system is concluded as

$$\begin{aligned}\dot{\sigma}_1 &= \beta\sigma_2 - \sigma_1 \\ \dot{\sigma}_2 &= (\sigma_2 + x_2)\sigma_3 - \sigma_2 - \beta\sigma_1 \\ \dot{\sigma}_3 &= -\sigma_3 - \sigma_4 - \sigma_2 x_2 - \sigma_2^2 \\ \dot{\sigma}_4 &= \sigma_1 + \sigma_3 + \alpha\sigma_4 + f_4 + u_4\end{aligned} \tag{48}$$

Step 4. In order to stabilize the $(\sigma_1-\sigma_2-\sigma_3-\sigma_4)$-system (48), the Lyapunov function V_4 is selected as

$$V_4 = V_1 + V_2 + V_3 + 0.5\sigma_4^2 \tag{49}$$

The derivative of V_4 is given by

$$\begin{aligned}\dot{V}_4 &= -\sigma_1^2 - \sigma_2^2 - \sigma_3^2 + \alpha\sigma_4^2 \\ &\quad + \sigma_1\sigma_4 + \sigma_4 f_4 + \sigma_4 u_4\end{aligned} \tag{50}$$

If $u_4 = -f_4 - \sigma_1 - (1+\alpha)\sigma_4$, then $\dot{V}_4 = -\sigma_1^2 - \sigma_2^2 - \sigma_3^2 - \sigma_4^2 < 0$, which implies that the $(\sigma_1-\sigma_2-\sigma_3-\sigma_4)$-system (48) is stable. Finally, the $(\sigma_1-\sigma_2-\sigma_3-\sigma_4)$-system is obtained as

$$\begin{aligned}\dot{\sigma}_1 &= \beta\sigma_2 - \sigma_1 \\ \dot{\sigma}_2 &= (\sigma_2 + x_2)\sigma_3 - \sigma_2 - \beta\sigma_1 \\ \dot{\sigma}_3 &= -\sigma_3 - \sigma_4 - \sigma_2 x_2 - \sigma_2^2 \\ \dot{\sigma}_4 &= \sigma_3 - \sigma_4\end{aligned} \tag{51}$$

Since $\dot{V}_4$ is negative definite, it follows that in the $(\sigma_1-\sigma_2-\sigma_3-\sigma_4)$ coordinates the equilibrium $(0,0,0,0)$ is stable. Consequently, by considering the $\sigma_1 = e_1$, $\sigma_2 = e_2 - \varphi_1(\sigma_1) = e_2$, $\sigma_3 = e_3 - \varphi_2(\sigma_1,\sigma_2) = e_3$ and $\sigma_4 = e_4 - \varphi_3(\sigma_1,\sigma_2,\sigma_3) = e_4$, one can obtain that synchronization errors e_1, e_2, e_3, e_4 converge to zero. In other words, the state variables of the slave system (36) track the ones of the master system (2).

6 Numerical Simulation Results

This section performs three numerical simulations to demonstrate the feasibility and effectiveness of the three designed backstepping control vectors to control and synchronize the hyperchaotic systems. The forth-order Runge-Kutta algorithm is applied to solve the differential equations with step size 0.0001 in all numerical simulations.

6.1 Numerical Simulation for Tracking Problem

The parameters of Wang system are considered as $a=10$, $b=40$, $c=2.5$, $d=10.6$, $k=1$ and $h=4$, to the Wang system exhibits hyperchaotic behaviors. The initial conditions of the controlled system (3) are defined as $\boldsymbol{x}(0) = \left[1,-1,-3,-4\right]^T$. Fig. 3 depicts that the output $x_1(t)$ can track the desired $x_d(t) = 5\sin 2t$ in the presence of control inputs vector $\boldsymbol{u} = [u_1, u_2, u_3, u_4]^T$. Notice that the controllers activated at $t = 5(\text{sec})$.

6.2 Synchronization Results of Two Identical Wang Systems

We have selected the parameters of the master and slave systems as $a=10$, $b=40$, $c=2.5$, $d=10.6$, $k=1$ and $h=4$. The initial conditions of the master and slave systems are assumed to be $\boldsymbol{x}_0 = [-2,1,1,-1]^T$ and $\boldsymbol{y}_0 = [7,-4,8,-4]^T$ respectively. Simulation results are shown in Figs. 4-6. The control inputs are activated at $t = 5(\text{sec})$. Fig. 4 depicts the state trajectories of master and slave systems. Figs. 5 and 6 show the time responses of synchronization errors and the Euclidean norm of errors vector respectively. It can be seen that the synchronization errors converge to zero after the controllers are activated at $t = 5(\text{sec})$.

6.3 Synchronization Results of Two Rössler and Wang Systems

In this subsection, the parameters of Rössler system are selected as $\alpha = 0.25$, $\beta = -0.5$ and $\gamma = 0.05$. More, the parameters of Wang system are considered as $a = 10$, $b = 40$, $c = 2.5$, $d = 10.6$, $k = 1$ and $h = 4$ respectively. The initial conditions of master and slave systems are assumed to be $\boldsymbol{x}_0 = [12, 5, -15, -1.2]^T$ and $\boldsymbol{y}_0 = [4, 8, -13, -2]^T$, respectively. Simulation results are illustrated in Figs. 7-9. The control inputs are activated at $t = 5(\text{sec})$. Fig. 7 shows the state trajectories of master and slave systems. Figs. 8 and 9 show the waveforms of synchronization errors and the Euclidean norm of errors vector. From the Fig. 8, it can be seen that the synchronization errors will converge to zero.

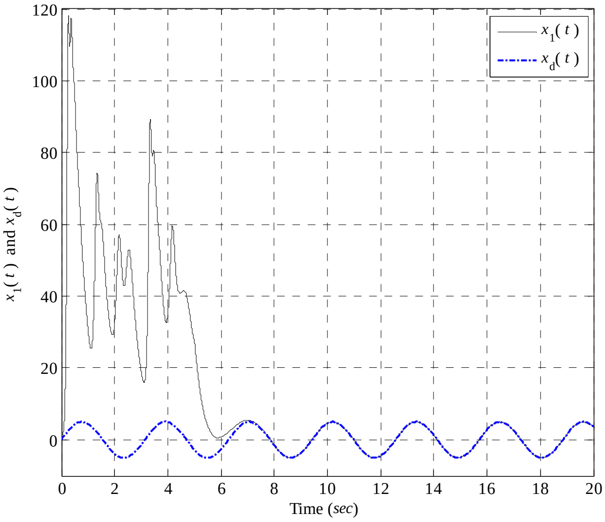

Fig. 3 Time response of $x_1(t)$ of Wang system which tracks the desired $x_d(t) = 5\sin 2t$.

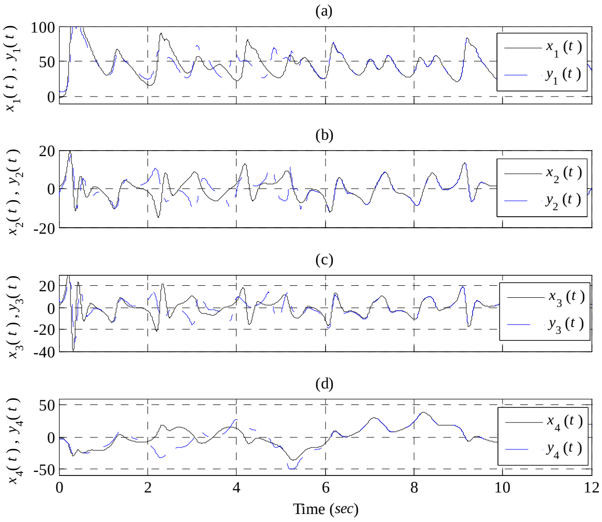

Fig. 4 Time responses of the state variables of master and slave Wang systems: (a) Time responses of $x_1(t), y_1(t)$, (b) Time responses of $x_2(t), y_2(t)$, (c) Time responses of $x_3(t), y_3(t)$, (d) Time responses of $x_4(t), y_4(t)$.

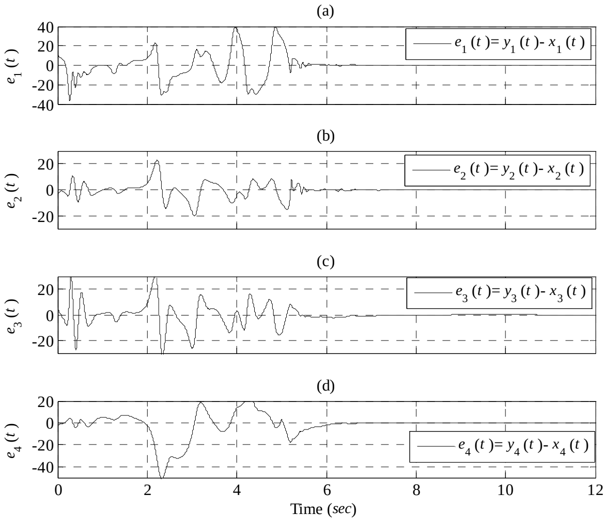

Fig. 5 Synchronization errors between the master and slave Wang systems: (a) Waveform of $e_1 = y_1 - x_1$, (b) Waveform of $e_2 = y_2 - x_2$, (c) Waveform of $e_3 = y_3 - x_3$, (d) Waveform of $e_4 = y_4 - x_4$.

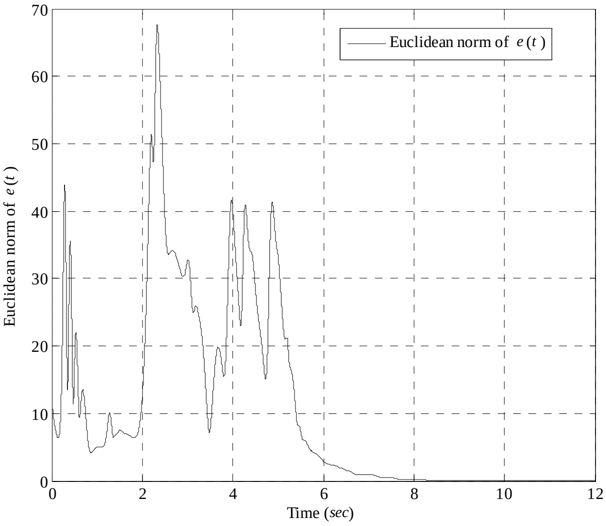

Fig. 6 Waveform of the Euclidean norm of synchronization errors vector.

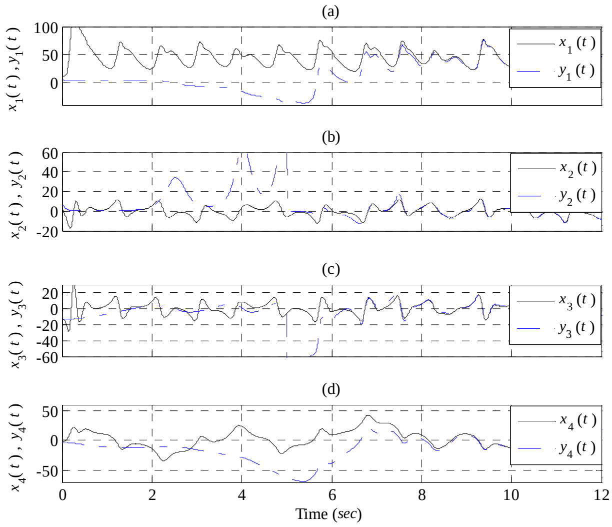

Fig. 7 Time responses of the state variables of Wang and Rössler systems: (a) Time responses of $x_1(t), y_1(t)$, (b) Time responses of $x_2(t), y_2(t)$, (c) Time responses of $x_3(t), y_3(t)$, (d) Time responses of $x_4(t), y_4(t)$.

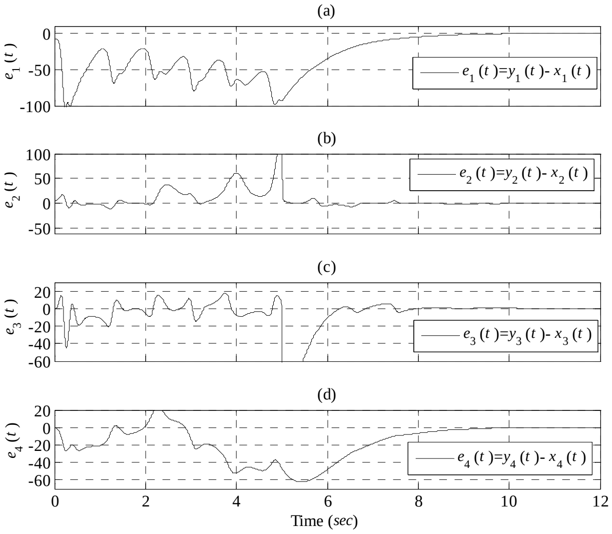

Fig. 8 Synchronization errors between the Wang and Rössler systems: (a) Waveform of $e_1 = y_1 - x_1$, (b) Waveform of $e_2 = y_2 - x_2$, (c) Waveform of $e_3 = y_3 - x_3$, (d) Waveform of $e_4 = y_4 - x_4$.

7 Conclusion

An approach for tracking and synchronization problems of hyperchaotic systems using active backstepping method has been proposed in this paper. The mentioned method has been applied to the hyperchaotic Rössler and Wang systems successfully. Three numerical simulations are used to demonstrate the effectiveness of the designed active backstepping controllers.

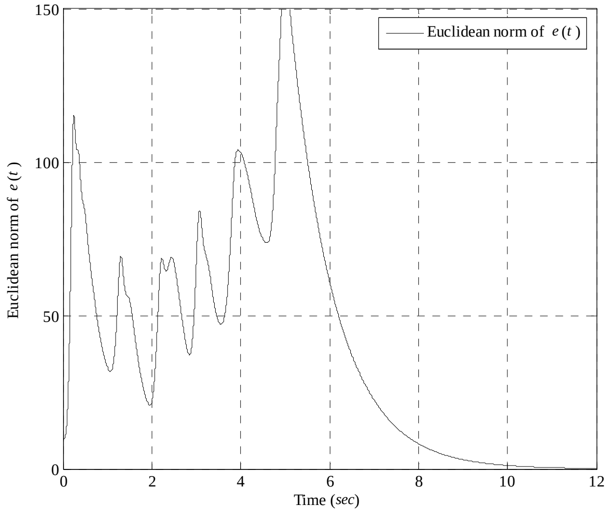

Fig. 9 Waveform of the Euclidean norm of synchronization errors vector.

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