Iranian Journal of Electrical and Electronic Engineering 04 (2019) 453–461

IUST Iran University of Science and Technology

Iranian Journal of Electrical and Electronic Engineering

Journal Homepage: ijeee.iust.ac.ir

Research Paper

A Novel Approach to Designing of Chattering-Free Sliding-Mode Control in Second-Order Discrete-Time Systems

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Abstract: In this paper, a chattering-free sliding-mode control is mainly proposed in a second-order discrete-time system. For achieving this purpose, firstly, a suitable control law would be derived by using the discrete-time Lyapunov stability theory and the sliding-mode concept. Then the input constraint is taken into account as a saturation function in the proposed control law. In order to guarantee the closed-loop system stability, a sufficient stability condition would be addressed in the presence of unstructured uncertainties. Hence the states of the discrete-time system are moved to a predefined sliding surface in a finite sampling time. Then the system states are asymptotically converged to the origin through the sliding line. The suggested SMC is successfully applied in two discrete-time systems (i.e. regulation and tracking problems). The effectiveness of the proposed method will be verified via numerical examples.

Keywords: Sliding-Mode Control, Discrete-Time System, Chattering-Free and Lyapunov Stability.

1 Introduction

THE sliding-mode control (SMC) is initially introduced in a typical second-order continuous-time system [1]. Then the SMC has been progressively interested by the other control engineers and researchers [2, 3]. Due to inherently robustness properties, the SMC has been increasingly applied in some uncertain industrial applications [4, 5]. Some recent versions of the SMC as well as terminal SMC [6, 7], modified terminal SMC [8], high order and integral SMC [9-11] have been popularly used in the present time.

The reaching and sliding phases are two main stages of the conventional SMC. The system states are quickly moved to a sliding surface in a finite-time at the reaching phase. Then the states would remain on the sliding surface [12]. In the continuous-time SMC, chattering phenomena is emerged due to the existence of the sign function in the control law [13].

Some control techniques have been suggested to reduce or remove the chattering phenomena [14-16]. For example, the sign term of the SMC may be approximated with a smoothed function [17, 18]. Then the chattering effect would be completely removed in the sliding stage but the reaching phase may not be accomplished in a finite-time [15].

Although the SMC is principally investigated in the continuous-time systems. But the concept of the SMC is also extended to the discrete-time systems [19, 20]. The system dynamic would be mathematically described with an algebraic equation in the discrete-time systems. Furthermore, they may have some excellent closed-loop properties as well as the existence of dead-beat response [21]. Thus the SMC concept can be also studied in the discrete-time systems.

Recently some SMC techniques have been suggested in the discrete-time systems [22-25]. Hence the discrete-time systems can be stabilized by designing a suitable SMC [26, 27]. In the discrete-time representation, a robot manipulator is stabilized via a second-order SMC combining with time-delay control [28]. An SMC is designed in the linear discrete-time system subject to input saturation [29]. A quasi SMC is suggested in the discrete-time systems with a bounded disturbance [30].

In order to guarantee some control properties like robust stability and asymptotic convergence, the implementations of the SMC in the discrete-time systems may have some control problems [31, 32]. Sometimes, in the discrete-time dynamical systems, the

Iranian Journal of Electrical and Electronic Engineering, 2019.
Paper first received 01 October 2018, revised 16 March 2019, and accepted 19 March 2019.
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SMC parameters may be optimized by using the mathematical algorithms [33]. From time to time, a switching type reaching law may have some benefits in the discrete-time SMC [34]. Based on the SMC concept, a second-order recursive reaching law may be found in the discrete-time systems [35].

Even though the conventional SMC is generally realized as a state feedback control law. But the output feedback SMC can be also designed in the discrete-time systems [36, 37]. Nowadays control schemes equipped with learning capabilities have been increasingly used in industrial applications. Hence the sliding-mode concept can be used to design a robust SMC in discrete-time control systems with learning abilities [38].

The uncertain terms may be taken into account in nonlinear discrete-time systems. Then an SMC could be designed in case of matched perturbations [39]. In some control problems, the control synthesis would be numerically solved by using linear matrix inequality (LMI). Hence, based on the LMI, the SMC could be designed in the discrete-time systems [40].

In the discrete-time systems, sometimes there may exist a response with the dead-beat property. This point motivates the author to develop a chattering-free SMC scheme in a second-order discrete-time system. This control method would have been very important when the control input is subjected to a hard constraint. For achieving this goal, a typical second-order system is firstly considered. Then a chattering-free SMC is derived in the discrete-time systems by using the well-known Lyapunov stability theory. Hence SMC scheme with the dead-beat property of the sliding surface has been interested in this paper. In order to handle the input constraint, some modifications would be made in the control sequences of the proposed method corresponding to the reaching phase of the SMC. Then the chattering effect is fully removed and also the input constraint would be handled by using the suggested procedure.

The rest of this paper is organized as follows: In the next section, the discrete-time SMC problem is formulated. In Section 3, a control law would be derived based on the sliding-mode idea. Then the system uncertainty would be investigated. In Section 4, two numerical examples are provided to compare the proposed SMC with the existed results. Some concluding remarks are finally placed in the last section.

2 Problem Formulation

Consider the following second order discrete-time system:

$$\begin{cases} \xi_1(k+1) = \xi_1(k) + \rho \tilde{f}_1(\boldsymbol{\xi}(k)) + \rho \tilde{g}_1(\boldsymbol{\xi}(k)) u(k) \\ \xi_2(k+1) = \xi_2(k) + \rho \tilde{f}_2(\boldsymbol{\xi}(k)) + \rho \tilde{g}_2(\boldsymbol{\xi}(k)) u(k) \end{cases} \quad \forall k \geq i \tag{1}$$

where ρ is a positive constant, $\boldsymbol{\xi}(k) = [\xi_1(k)\ \xi_2(k)]^T \in \mathbb{R}^2$ denotes the system states and the scalar $u(k)$ would be the control input of the discrete-time system (1).

It is shown that the discrete-time system (1) can be found by determining the Euler's backward approximation of the following continuous-time system:

$$\begin{cases} \dot{\xi}_1(t) = \tilde{f}_1(\boldsymbol{\xi}(t)) + \tilde{g}_1(\boldsymbol{\xi}(t)) u(t) \\ \dot{\xi}_2(t) = \tilde{f}_2(\boldsymbol{\xi}(t)) + \tilde{g}_2(\boldsymbol{\xi}(t)) u(t) \end{cases} \tag{2}$$

Assumption 1: For taking of simplicity purpose, it is assumed that there exist transformation $x_1 = \Gamma_1(\xi_1, \xi_2)$ and $x_2 = \Gamma_2(\xi_1, \xi_2)$ such that the discrete-time system (1) would be written as follows:

$$\begin{cases} x_1(k+1) = x_1(k) + \rho x_2(k) \\ x_2(k+1) = x_2(k) + \rho f(\boldsymbol{x}(k)) + \rho g(\boldsymbol{x}(k)) u(k) \end{cases} \quad \forall k \geq i \tag{3}$$

where $\boldsymbol{x}(k) = [x_1(k)\ x_2(k)]^T \in \mathbb{R}^2$ denotes the states of the discrete-time system (3).

The initial time is denoted by the symbol i. It is trivial that without loss of generality the initial time i can be set to zero and the constant $\rho > 0$ can be treated as the sampling-time T_s.

Assumption 2: The vector fields $g(.)\colon \mathbb{R}^2 \to \mathbb{R}$ and $f(.)\colon \mathbb{R}^2 \to \mathbb{R}$ may be described some well-defined and known functions. Furthermore the term $g(.)$ is always a non-zero function $g(\boldsymbol{\eta}) \neq 0$, $\boldsymbol{\eta} \in \mathbb{R}^2$. It is also assumed that the discrete-time system (3) is stabilizable under the saturated input sequence like $|u(k)| \leq \bar{u}$.

In order to design a chattering-free control law, the discrete-time system (3) is taken into account in this paper. Let select the sliding manifold $s(k)$ as follows:

$$s(k) = x_2(k) + \alpha x_1(k), \quad 0 < \alpha < \frac{2}{\rho} \tag{4}$$

In this case, the sliding surface $s(k)$ would be really a straight line with a negative slope. The mathematical condition $0 < \alpha < 2/\rho$ would preserve the stability of the closed loop system. It is desired to design some sequences of the control input $u(k)$, $k \geq i$ such that the state vector $x(k)$ of the discrete-time system (3) can reach to the sliding surface $s(k) = 0$. Then it would remain on the straight line $x_2 = -\alpha x_1$. Therefore the closed loop would be asymptotically stable while the system states would be kept in the predefined sliding surface. In the next section, the SMC scheme will be investigated in the discrete-time system (3).

3 Main Result

In this section, the SMC scheme would be derived in the discrete-time system (3) by using of the Lyapunov stability theory. Then the robustness of the proposed method would be investigated in presence of the system uncertainty. Hence a chattering-free control law will be addressed in the next proposition.

Proposition 1: Consider the second order discrete-time system (3) with Assumption 2. The following control sequence $u(k)$ is a chattering-free control law:

$$u(k) = -\frac{s(k) + \rho f(\boldsymbol{x}(k)) + \alpha\rho x_2(k)}{\rho g(\boldsymbol{x}(k))} \quad \forall k \geq i \tag{5}$$

then the control input (5) would move the states of discrete-time system (3) to the sliding surface (4) in a finite-time and the closed loop system would be asymptotically stable.

Proof: A quadratic Lyapunov function $V(k) = s^2(k)$ can be taken into account. The difference of the Lyapunov function $\Delta V(k) = V(k+1) - V(k)$ would be obtained as follows:

$$\Delta V(k) = (s(k+1) + s(k))(s(k+1) - s(k)) \tag{6}$$

Then

$$\begin{aligned}\Delta V(k) &= (x_2(k+1) + \alpha x_1(k+1) + x_2(k) + \alpha x_1(k)) \\ &\quad \times (x_2(k+1) + \alpha x_1(k+1) - x_2(k) - \alpha x_1(k))\end{aligned} \tag{7}$$

The equation (7) can be written as the following:

$$\begin{aligned}\Delta V(k) &= (2s(k) + \rho f(\boldsymbol{x}(k)) + \rho g(\boldsymbol{x}(k))u(k) \\ &\quad + \alpha\rho x_2(k)) \times (\rho f(\boldsymbol{x}(k)) + \rho g(\boldsymbol{x}(k))u(k) \\ &\quad + \alpha\rho x_2(k))\end{aligned} \tag{8}$$

Then the condition (8) may be simplified as follows:

$$\begin{aligned}\Delta V(k) &= (s(k) + \rho f(\boldsymbol{x}(k)) + \rho g(\boldsymbol{x}(k))u(k) \\ &\quad + \alpha\rho x_2(k))^2 - s^2(k)\end{aligned} \tag{9}$$

The control input $u(k)$ can be deliberately selected such a way that the first term of the equation (9) is exactly zero. Then

$$s(k) + \rho f(\boldsymbol{x}(k)) + \rho g(\boldsymbol{x}(k))u(k) + \alpha\rho x_2(k) = 0 \quad \forall k \geq i \tag{10}$$

Therefore the control input $u(k)$ can be analytically determined as follows:

$$u(k) = -\frac{s(k) + \rho f(\boldsymbol{x}(k)) + \alpha\rho x_2(k)}{\rho g(\boldsymbol{x}(k))} \quad \forall k \geq i \tag{11}$$

It completes the proof.

The effect of the bounded control input is mainly investigated in the subsequent. Let define the variables $q_r(k)$ and $q_s(k)$ as the following:

$$q_r(k) \overset{\text{def}}{=} -\frac{s(k) + \rho f(\boldsymbol{x}(k)) + \alpha\rho x_2(k)}{\rho g(\boldsymbol{x}(k))} \tag{12}$$

$$q_s(k) \overset{\text{def}}{=} -\frac{f(\boldsymbol{x}(k)) + \alpha x_2(k)}{g(\boldsymbol{x}(k))} \tag{13}$$

The subscripts r and s may denote the reaching and sliding phases of the SMC, respectively.

Hence the signal $q_r(i)$ is applied into the discrete-time system (3) in the reaching phase. Similarly in the sliding phase (i.e. $k \geq i + 1$), the control signal $q_s(k)$ is used. Then the Lyapunov function difference is found as $\Delta V(k) = -s^2(k)$. It can be shown that the sliding surface dynamic is determined as follows:

$$s(k) = 0 \quad \forall k \geq i+1 \tag{14}$$

Then the system states $x(k)$ would reach to the sliding surface $s(k) = 0$ in a single-step. Therefore the dead-beat property would exist in the sliding variable $s(k)$. In other word, we have the following condition:

$$\text{If } s(i) \neq 0 \text{ then } s(i+1) = 0. \tag{15}$$

It can be checked that the closed loop system would be reduced to the following equation:

$$x_1(k+1) = (1 - \rho\alpha)x_1(k) + \rho s(k) \tag{16}$$

Therefore it is clear that the closed loop system would be asymptotically stabilized when the inequality $0 < \alpha < 2/\rho$ is satisfied. Then the reaching phase is completely accomplished in a single step by the proposed method. This method can be easily extended to multi steps cases. Hence the system state $x(k)$ could reach to the sliding surface in multiple steps (namely N-step) to ensure the condition $s(k) = 0$. In this case, it suffices to select the control input $u(k)$ as the following:

$$u(k) = \begin{cases} \text{arbitary} & i \leq k < N + i - 1 \\ q_r(k) & k = N + i - 1 \\ q_s(k) & k > N + i - 1 \end{cases} \tag{17}$$

Hence the first N-step of control law $u(k)$ would be some arbitrary signals. Someone may manually select the extremum control effort in this case. The proposed SMC under actuator saturation is discussed in the next remark.

Remark 1: The suggested strategy could be very helpful when the input signal $u(k)$ is imposed to a hard constraint like $|u(k)| \leq \bar{u}$. In this case, the control signal $sat(u(k))$ could be used in the discrete-time system (3). In the reaching phase (i.e. $k \leq N + i - 1$), some control signals like $\bar{u}\, sign(u(k))$ are applied into the discrete-time system (3). It is assumed that the dynamical system (3) is stabilizable with constrained input. Hence the input $\bar{u}\, sign(u(k))$ would not destabilized the closed loop system. The constrained input would be applied into the discrete-time system (3) in order to enforce the sliding condition $s(k) = 0$. Then the control signal

$q_r(N+i-1)$ is used to move the system states into the sliding line $s(k) = 0$. Then the system states would reach to the sliding surface in some finite steps (i.e. $N+i-1$ samplig times) while the system input is imposed with such an input constraint. In the sliding phase, the signal $q_s(k)$ is used to preserve the system states on the sliding surface.

Remark 2: The tracking problem can be investigated in the discrete-time system (3). Hence the following reference model is considered:

$$\begin{cases} x_{1m}(k+1) = x_{1m}(k) + \rho x_{2m}(k) \\ x_{2m}(k+1) = x_{2m}(k) + \rho h(\boldsymbol{x_m}(k), r(k)) \end{cases} \forall k \geq i \tag{18}$$

where $\boldsymbol{x}_m(k) = [x_{1m}(k)\ x_{2m}(k)]^T \in \mathbb{R}^2$ and $r(k) \in \mathbb{R}$ denotes a reference signal. It is desired the system states $x(k)$ can track the reference signal $x_m(k)$. Then the tracking errors are defined as follows:

$$\begin{cases} e_1(k) = x_1(k) - x_{1m}(k) \\ e_2(k) = x_2(k) - x_{2m}(k) \end{cases} \tag{19}$$

It can be shown that the error dynamic would be written as the following form:

$$\begin{cases} e_1(k+1) = e_1(k) + \rho e_2(k) \\ e_2(k+1) = e_2(k) + \rho f(\boldsymbol{x}(k)) + \rho g(\boldsymbol{x}(k)) u(k) \\ \qquad - \rho h(\boldsymbol{x_m}(k), r(k)) \end{cases} \tag{20}$$

Let define the sliding surface as $s(k) = e_2(k) + \alpha e_1(k)$, $0 < \alpha < 2/\rho$. Then a similar design procedure, which addressed in Proposition 1, can be used to find the chattering-free SMC. Thus the control input $u(k)$ is determined as follows:

$$u(k) = -\frac{s(k) + \rho f(\boldsymbol{x}(k)) + \alpha\rho e_2(k)}{\rho g(\boldsymbol{x}(k))} - \frac{-\rho h(\boldsymbol{x_m}(k), r(k))}{\rho g(\boldsymbol{x}(k))} \ \forall k \geq i \tag{21}$$

Therefore the control law (21) would be a chattering-free control law and the dynamical system (3) can track the reference model (18). Furthermore the sliding variable $s(k) = e_2(k) + \alpha e_1(k)$ would be stabilized in a finite-time.

Remark 3: The proposed idea can be extended to the following discrete-time MIMO system:

$$\begin{cases} \boldsymbol{x}_1(k+1) = \boldsymbol{x}_1(k) + \mathrm{P}\boldsymbol{x}_2(k) \\ \boldsymbol{x}_2(k+1) = \boldsymbol{x}_2(k) + \mathrm{P}F(\boldsymbol{x}(k)) + \mathrm{P}G(\boldsymbol{x}(k)) u(k) \end{cases} \forall k \geq i \tag{22}$$

where $\boldsymbol{P} = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_m)$ is a diagonal matrix, $\boldsymbol{x}(k) = [\boldsymbol{x}_1(k)\ \boldsymbol{x}_2(k)]^T$, $\boldsymbol{x}_1(k) \in \mathbb{R}^m$, $x_2(k) \in \mathbb{R}^m$ would denote the state vectors and $u(k) \in \mathbb{R}^m$ would be the input vector of the discrete-time system (22). Furthermore the vector fields $\boldsymbol{G}(.):\mathbb{R}^{2m} \to \mathbb{R}^{m\times m}$, where $\boldsymbol{G}(\boldsymbol{\eta})$ is invertible $\forall \boldsymbol{\eta} \in \mathbb{R}^{2m}$ and $\boldsymbol{F}(.):\mathbb{R}^{2m} \to \mathbb{R}^m$ may be described some well-defined and known functions. In this case, the sliding vector can be similarly defined as $\boldsymbol{s}(k) = \boldsymbol{x}_2(k) + \boldsymbol{A}\boldsymbol{x}_1(k)$ where $\boldsymbol{A} = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m)$. It is assumed the eigenvalues of the matrix I_m-PA would be placed in the unity circle (i.e. $0 < \alpha_i < 2/\rho_i$). Then the proposed SMC law may be obtained as follows:

$$\boldsymbol{u}(k) = -\boldsymbol{G}^{-1}(\boldsymbol{x}(k))\left(\boldsymbol{P}^{-1}\boldsymbol{s}(k) + \boldsymbol{F}(\boldsymbol{x}(k)) + \boldsymbol{A}\boldsymbol{x}_2(k)\right) \quad \forall k \geq i \tag{23}$$

The input vector (23) would be a chattering-free control law and it can stabilize the sliding vector $\boldsymbol{s}(k)$ in a finite-time. Furthermore the closed loop system would be asymptotically stable.

Remark 4: The proposed idea is mainly presented in Proposition 1. It is not difficult to check that it can be extended to the high order discrete-time system in strict feedback form. For this purpose, let consider the following dynamical system:

$$\begin{cases} x_j(k+1) = x_j(k) + \rho x_{j+1}(k),\ j = 1, 2, \ldots, n-1 \\ x_n(k+1) = x_n(k) + \rho f(\boldsymbol{x}(k)) + \rho g(\boldsymbol{x}(k)) u(k) \end{cases} \forall k \geq i \tag{24}$$

$$s_j(k) = x_{j+1}(k) + \alpha_j x_j(k),\ 0 < \alpha_j < \frac{2}{\rho}, \quad j = 1, 2, \ldots, n-1 \tag{25}$$

The asymptotic stability of the closed loop system can be preserved by assuming the mathematical condition $0 < \alpha_j < 2/\rho$. Therefore a similar procedure can be used to designing of a chattering-free SMC in the discrete-time system (24).

Sometimes the nonlinear functions $f(.)$ and $g(.)$ may be subjected to the unstructured uncertainty. Hence the proposed control law (5) is investigated in presence of the unstructured uncertainties in the subsequent.

Assumption 3: The terms $f(.)$ and $g(.)$ are some uncertain functions. The uncertainty of the nonlinear terms $f(.)$ and $g(.)$ are assumed to be in additive and multiplicative forms respectively as follows:

$$f(\xi) = \hat{f}(\xi) + \beta_f \delta f(\xi) \tag{26}$$

$$g(\xi) = \hat{g}(\xi)\left(1 + \beta_g \delta g(\xi)\right) \tag{27}$$

where $\hat{f}(.)$ and $\hat{g}(.)$ are exactly known functions. Furthermore the term $\hat{g}(.)$ may always a non-zero function. The terms $\delta f(.)$ and $\delta g(.)$ are some unknown terms which hold on the following conditions:

$$|\delta f(\xi)| \leq 1 \tag{28}$$

$$|\delta g(\xi)| \leq 1 \tag{29}$$

The positive constants β_f and β_g are some known weights which satisfy the following inequality:

$$\rho\beta_f + \beta_g \left| s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k) \right| \le \left| s(k) \right| \tag{30}$$

Therefore the uncertainty of the nonlinear terms $f(.)$ and $g(.)$ would be limited to the inequality (30). Thus the equation (30) may determine the effect of the system uncertainties on the closed loop stability. Hence in the discrete-time system (3) without the uncertain terms, someone may have $\beta_f = 0$ and $\beta_g = 0$, then the inequality (30) would be inherently fulfilled.

Proposition 2: Consider the second order discrete-time uncertain system (3) with Assumptions 2 and 3. The following control input $u(k)$ would be a chattering-free control scheme and it can stabilize the closed loop system.

$$u(k) = -\frac{s(k) + \rho f(\boldsymbol{x}(k)) + \alpha\rho x_2(k)}{\rho g(\boldsymbol{x}(k))} \ \forall\ k \ge i \tag{31}$$

Proof: The Lyapunov function $V(k) = s^2(k)$ may be considered again. Then with the unstructured uncertain terms $\delta f(.)$ and $\delta g(.)$, the difference of the Lyapunov function $\Delta V(k)$ would be written as:

$$\begin{aligned}\Delta V(k) = \Big(& s(k) + \rho\left(\hat{f}(\boldsymbol{x}(k)) + \beta_f \delta f(\boldsymbol{x}(k))\right) \\ & + \rho\left(\hat{g}(\boldsymbol{x}(k))\left(1 + \beta_g \delta g(\boldsymbol{x}(k))\right)\right) u(k) + \alpha\rho x_2(k)\Big)^2 \\ & - s^2(k)\end{aligned} \tag{32}$$

The equation (32) can be rewritten as follows:

$$\begin{aligned}\Delta V(k) = \Big(& s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \rho\beta_f \delta f(\boldsymbol{x}(k)) \\ & - \left(1 + \beta_g \delta g(\boldsymbol{x}(k))\right)\left(s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right) \\ & + \alpha\rho x_2(k) \Big)^2 - s^2(k)\end{aligned} \tag{33}$$

Then

$$\begin{aligned}\Delta V(k) = \Big(& s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \rho\beta_f \delta f(\boldsymbol{x}(k)) \\ & - \left(s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right) \\ & - \beta_g \delta g(\boldsymbol{x}(k))\left(s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right) \\ & + \alpha\rho x_2(k)\Big)^2 - s^2(k)\end{aligned} \tag{34}$$

It can be simplified as follows:

$$\begin{aligned}\Delta V(k) = \Big(& \rho\beta_f \delta f(\boldsymbol{x}(k)) \\ & - \beta_g \delta g(\boldsymbol{x}(k))\left(s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right)\Big)^2 \\ & - s^2(k)\end{aligned} \tag{35}$$

Assumption 3 implies that the first term of (35) can be bounded above as:

$$\begin{aligned}& \rho\beta_f \delta f(\boldsymbol{x}(k)) - \beta_g \delta g(\boldsymbol{x}(k))\left(s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right) \le \\ & \rho\beta_f \left|\delta f(\boldsymbol{x}(k))\right| + \beta_g \left|\delta g(\boldsymbol{x}(k))\right| . \left|s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right| \le \\ & \rho\beta_f + \beta_g \left|s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right|\end{aligned} \tag{36}$$

Thus the inequality (35) may be written as:

$$\begin{aligned}\Delta V(k) = & \left(\rho\beta_f + \beta_g \left|s(k) + \rho \hat{f}(\boldsymbol{x}(k)) + \alpha\rho x_2(k)\right|\right)^2 \\ & - s^2(k)\end{aligned} \tag{37}$$

By applying (30), the difference of the Lyapunov function $\Delta V(k)$ would be a negative definite function. It completes the proof.

In comparing with the existed method, the feasibility of the proposed method is investigated with some numerical examples in the next section.

4 Simulation Results

Example 1: Consider the van der pol equation which discretized via the Euler's backward approximation as follows:

$$\begin{cases} x_1(k+1) = x_1(k) + T_s x_2(k) \\ x_2(k+1) = x_2(k) + \mu T_s\left(1 - x_1^2(k)\right) x_2(k) \\ \qquad - T_s x_1(k) + 5 T_s u(k) \end{cases} \tag{38}$$

where $\mu = 1$ and sampling time T_s is set as 0.1 seconds. The control input $u(k)$ is a bounded as $|u(k)| \le 1$. For sake of comparison, two other SMCs are designed with the recent and conventional results. By applying the existed methods, the SMC law would be analytically found as follows [13]:

$$u(k) = -\frac{f(\boldsymbol{x}(k)) + \alpha x_2(k) + \gamma sgn(s(k))}{g(\boldsymbol{x}(k))} \tag{39}$$

where the design parameter γ would be a positive constant. Then the control law (39) can be simplified as follows:

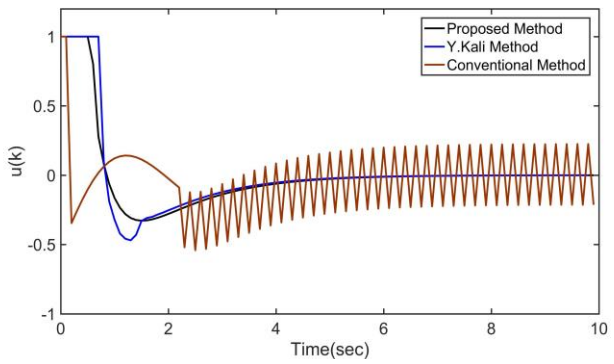

Fig. 1 The applied control input $u(k)$.

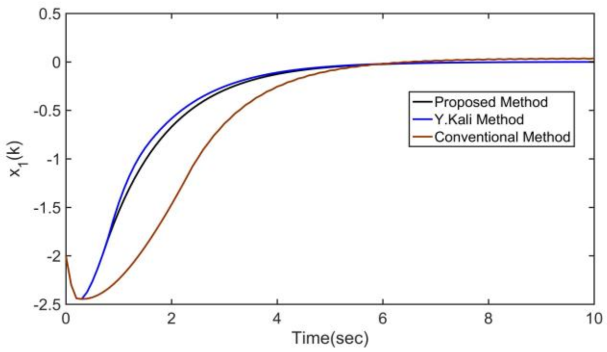

Fig. 2 The states $x_1(k)$ of the van der pol system.

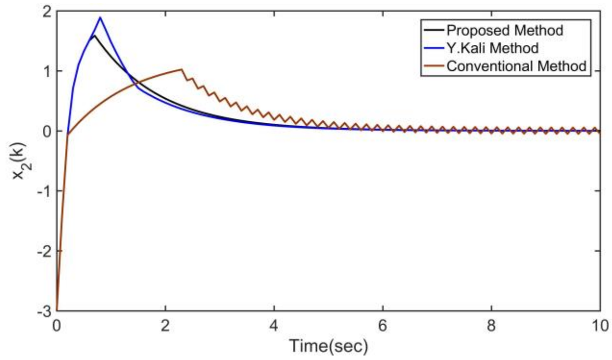

Fig. 3 The states $x_2(k)$ of the van der pol system.

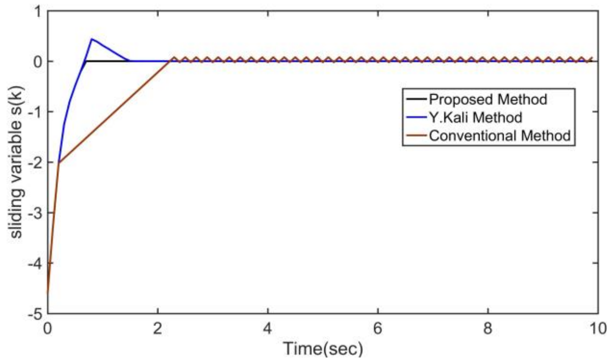

Fig. 4 The sliding variable $s(k)$.

$$u(k)=\frac{1}{5}\Big(x_1(k)-(1+\alpha)x_2(k)+x_1^2(k)x_2(k) -\gamma sgn(s(k))\Big) \quad (40)$$

The control law (40) is implemented with $\gamma = 1$ in this example. The proposed control policy is also compared with a recent SMC equipped with time-delay control which addressed by Y. Kali [28]. Hence the control $u(k)$ is generated with the following difference equation [28]:

$$u(k)=u(k-1)-\frac{1}{5}\Big(f(x(k))-f(x(k-1)) +\frac{1}{T_s}(x_2(k)-x_2(k-1))+\eta(k)\Big) \quad (41)$$

where

$$\begin{cases} f(x(k))=x_2(k)-x_1(k)-x_2(k)x_1^2(k) \\ \eta(k)=\alpha x_2(k)+\frac{\beta+1}{T_s}s(k)-\frac{1}{T_s}\varphi(k) \\ \varphi(k)=\varphi(k-1)-\gamma T_s sgn(\sigma(k)),\ \gamma>0 \\ \sigma(k)=s(k)+\beta s(k-1),\ 0<\beta<1 \\ s(k)=x_2(k)+\alpha x_1(k),\ 0<\alpha<1 \end{cases}$$

In the numerical simulation, the parameters of the SMC [28] are chosen as $\alpha = 0.8$, $\beta = 0.5$, and $\gamma = 1$. It is expected that the control sequence $u(k) = 0.2$ is applied to the van de pol system (38) about the origin. This point can be verified in the simulation results. By using of the proposed approach, the control sequences $q_r(k)$ and $q_s(k)$ can be computed as the following:

$$\begin{cases} q_r(k)=\left(1-\frac{\alpha}{T_s}\right)x_1(k)-\left(1+\alpha+\frac{1}{T_s}\right)x_2(k) +x_1^2(k)x_2(k) \\ q_s(k)=x_1(k)-(1+\alpha)x_2(k)+x_1^2(k)x_2(k) \end{cases} \quad (42)$$

The initial conditions of the van der pol system (38) are selected as $x_1(0) = -2$ and $x_2(0) = -3$. The slope of sliding line is set as $\alpha = 0.8$ in three SMC laws (i.e. equations (39), (41), and (42)). This example is numerically simulated by using of the control laws (39), (41), and (42) in the nonlinear discrete-time system (38) and the results are shown in Figs. 1-5. The suggested control law (42) would be a state feedback control policy without any sign function. This point would be a key difference of the proposed control law with the well-known continuous-time SMC. The applied control signal $u(k)$ can be depicted in Fig. 1. It is seen that no sign function is existed in the proposed control law (42). Hence, no chattering phenomena would be emerged in the control signal $u(k)$. The states of the discrete-time system (38) are shown in Figs. 2 and 3. In the van der pol system, the sliding variable $s(k)$ is shown in Fig. 4.

As seen in Fig. 5, the states of the van der pol system (38) are reached to the sliding line $s(k) = 0$ within $N = 8$ steps by applying the proposed method. In the van der pol system (38), the reaching phase is accomplished in 15 steps by the Y. Kali SMC. Then they would remain in the sliding line. The superiority of suggested method with comparing to the existed SMCs can be shown in the simulation results. It is seen that the proposed method would be a chattering-free SMC in the

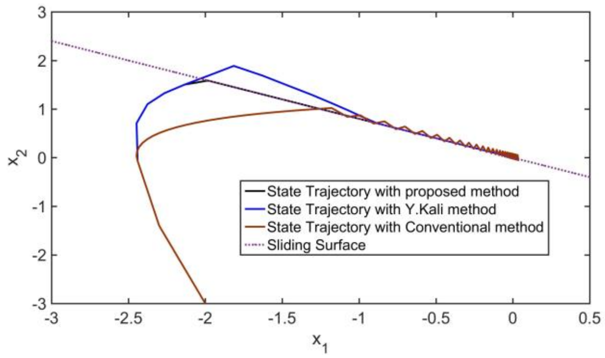

Fig. 5 The states trajectory (x_1x_2) in the van der pol system.

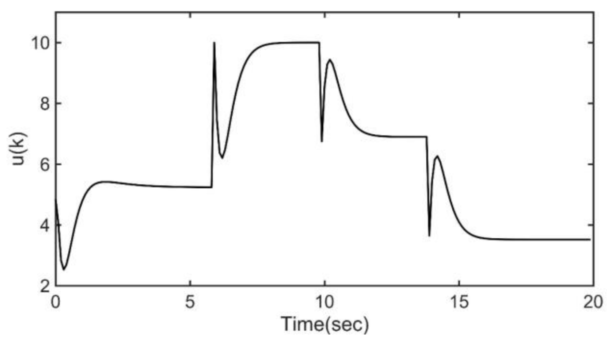

Fig. 6 The applied torque $u(k)$ into the inverted pendulum.

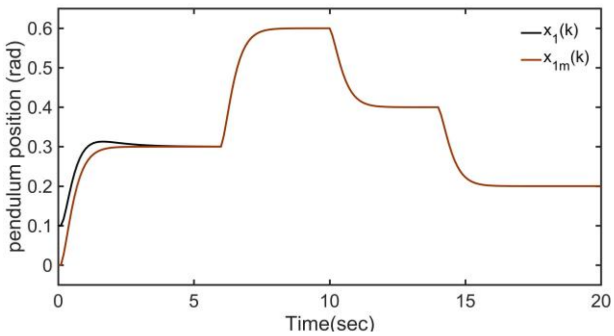

Fig. 7 The positions $x_1(k)$ and $x_{1m}(k)$ in the inverted pendulum.

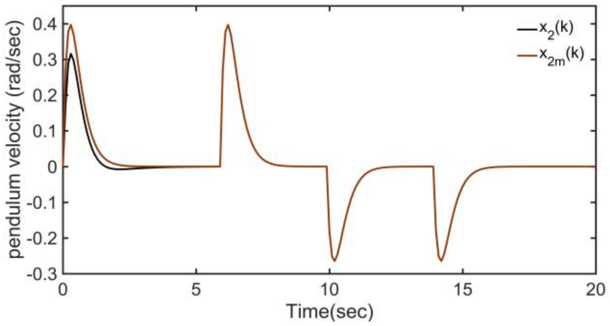

Fig. 8 The velocities $x_2(k)$ and $x_{2m}(k)$ in the inverted pendulum.

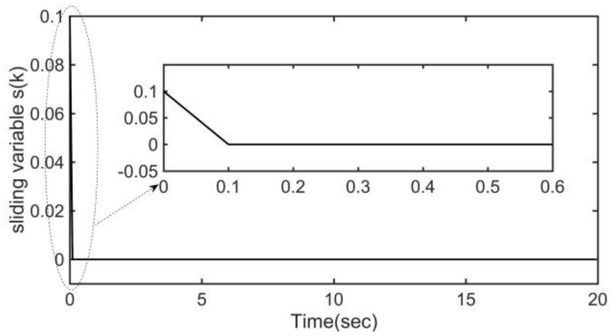

Fig. 9 The sliding variable $s(k)$.

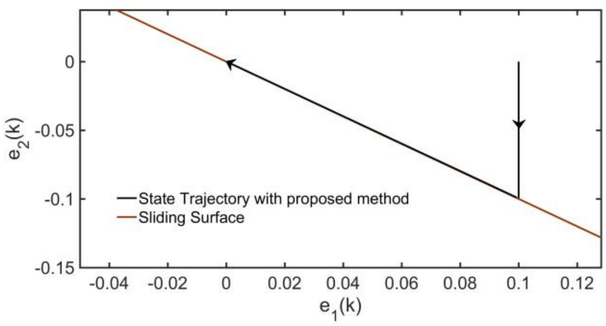

Fig. 10 The error trajectory (e_1e_2).

discrete-time dynamical system while the actuator saturation is taken into account.

Example 2: Consider the following discrete-time inverted pendulum:

$$\begin{cases} x_1(k+1) = x_1(k) + T_s x_2(k) \\ x_2(k+1) = x_2(k) - \dfrac{gT_s}{l}\sin(x_1(k)) \\ \qquad - \dfrac{kT_s}{m} x_2(k) + \dfrac{T_s}{ml^2} u(k) \end{cases} \tag{43}$$

where $k = 1$N/m, $g = 9.81$ N/kg, $m = 2$kg, and $l = 1$m.

It is desired to design a control policy $u(k)$ in order to track the following reference model:

$$\begin{cases} x_{1m}(k+1) = x_{1m}(k) + T_s x_{2m}(k) \\ x_{2m}(k+1) = x_{2m}(k) - 9T_s x_{1m}(k) - 6T_s x_{2m}(k) \\ \qquad +18T_s r(k) \end{cases} \tag{44}$$

where the reference signal $r(k)$ is selected as follows:

$$r(k) = \begin{cases} 0.3\ ,\ 0 \le kT_s < 6 \\ 0.6\ ,\ 4 \le kT_s < 10 \\ 0.4\ ,\ 10 \le kT_s < 14 \\ 0.2\ ,\ 14 \le kT_s < 20 \end{cases}$$

The sampling-time T_s sets as 0.1 seconds. The slope of the sliding line is selected as $\alpha = 1$.

The initial conditions of the discrete-time systems (43) and (44) are chosen as $x_1(0) = 0.1$rad, $x_2(0) = 0$rad/sec, $x_{1m}(0) = 0$rad, $x_{2m}(0) = 0$rad/sec.

By applying of Remark 2, the control input $u(k)$ is determined as follows:

$$\begin{cases} u(k) = -20s(k) + 19.62\sin(x_1(k)) - x_2(k) \\ \qquad -18x_{1m}(k) - 10x_{2m}(k) + 18r(k) \\ s(k) = x_2(k) + x_1(k) + x_{2m}(k) + x_{1m}(k) \end{cases} \tag{45}$$

The applied torque into the inverted pendulum $u(k)$ is shown in Fig. 6. The position and velocity of the inverted pendulum (i.e. the states of the discrete-time system (43)) are depicted in Figs. 7 and 8. It is clear that the reference system (44) has been tracked with the proposed control law within 5 seconds. The sliding variable $s(k)$ and error trajectory are shown in Figs. 9 and 10. It is seen that the sliding variable $s(k)$ reaches to the origin in a finite-time.

5 Conclusion

A chattering-free sliding-mode control has been mainly proposed in the discrete-time dynamical systems with considering of the bounded input. For this purpose, a sliding-mode control law is analytically derived in a typical second order discrete-time system. It is shown that the system states are quickly moved to a predefined sliding surface in some finite sampling times. Then the system states are guided to the origin. The effect of the system uncertainty is also investigated in order to design a robust SMC. Two numerical examples are provided to show the effectiveness of the proposed method in comparing with the existed results.

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