A Study on Properties of Dempster-Shafer Theory to Probability Theory Transformations

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Abstract: In this paper, five conditions that have been proposed by Cobb and Shenoy are studied for nine different mappings from the Dempster-Shafer theory to the probability theory. After comparing these mappings, one of the considerable results indicates that none of the mappings satisfies the condition of invariance with respect to the marginalization process. In more details, the main reason for this defect is that the classic projection process in DST loses some probabilistic information. For solving this problem, the modified pignistic probability and the modified normalized plausibility transformation which are invariance under marginalization are proposed. The two modified mappings are utilized in two ambiguity measures in Dempster-Shafer theory. Then, similar to the mutual information in the probability theory, these measures are used for computing the dependency of the variables of a social bliss problem on the person's happiness value.

Keywords: Ambiguity, Dempster-Shafer Theory, DST to PT Transformation, Invariance with Respect to the Marginalization Process, Normalized Plausibility Transformation, Pignistic Probability.

1 Introduction

The Dempster-Shafer Theory (DST) and the Probability Theory (PT) are two theories that have been used for modeling uncertain data. In each theory, the combination and the marginalization rules are utilized for various applications. The main different of these two theories is that the Dempster-Shafer theory includes probability theory as well as set theory. In other words, in the Dempster-Shafer theory, the Basic Probability Assignment (BPA) is applied to assign masses to a subset of the frame of discernments while in the probability theory, the Probability Density Function (PDF) assigns values to the singleton members. The problems arise when we want to make a decision in DST. Therefore, the BPA in DST should be transformed to the probability density function in PT.

Some of these mappings are $PrPl_m$, PlP_m, $PrBel_m$, $PraPl_m$, and $PrHyb_m$ that proposed by Sudeno [1], $CuzzP_m$ as defined by Cuzzolin [2], the pignistic probability ($BetP_m$) as defined by Philippe Smets [3-6], and $DSmP_\epsilon$ as proposed by Smarandache *et al.* [7]. Besides the issue of decision making, these mappings are utilized in the aggregate uncertainty measures for computing the amounts of information in DST.

Smarandache *et al.* used the Probabilistic Information Content (PIC) as a measure of uncertainty for several probabilistic mappings from the DST to the PT. After comparing the mappings, they concluded that some mappings have conceptual problems and cannot produce acceptable results [7]. Klir *et al.* proposed AU as an aggregate uncertainty measure to compute the amounts of ambiguity in DST [8]. AU was defined based on another DST to PT mapping is denoted by $P_{6.1}$ as it is defined by the optimization process of the algorithm 6.1 [9-12].

Jousselme *et al.* used the pignistic probability to measure ambiguity (AM) in Dempster-Shafer theory [13]. But, AM dose not satisfy the subadditivity condition which is one of the necessary properties of an aggregate uncertainty measure. The cause of this defect which was found by Klir in [14] was the dependency of the pignistic probability on the marginalization process.

Another application of these mappings is transforming a Dempster-Shafer network into a Bayesian network. In several articles, Cobb and Shenoy have compared two mappings, the pignistic probability and the normalized plausibility transformation [15-18]. They proposed five conditions should be satisfied by a DST to PT transformation. These conditions are: *Invariance with respect to marginalization, Invariance with respect to combination, Idempotency, Unique most probable state, and Non-Unique most probable state.*

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The aim of this paper is to study the conditions mentioned by Cobb and Shenoy for nine mappings namely, $PrPl_m$, PlP_m, $PrBel_m$, $PraPl_m$, $PrHyb_m$, $CuzzP_m$, $BetP_m$, $DSmP_\epsilon$, and $P_{6.1}$. Furthermore, some considerable results are obtained. Another important matter which was addressed here is the invariance with respect to marginalization property of two mappings, the pignistic probability and the normalized plausibility transformation. The key of the *Invariance with respect to marginalization* problem of $BetP_m$ and PlP_m is in the projection and marginalization processes in DST. It can be shown that in the projection procedure, some probabilistic information is lost. If this data can be retained, the problem will be overcome. Based on this, a new set is presented which can store the probabilistic information lost in the projection process, and subsequently, the marginalization formula is slightly changed. Then, the modified pignistic probability and the modified normalized plausibility transformation are proposed that are invariant under the marginalization process.

The outline of this paper is as followings: In Section 2, we describe some necessary theoretical concepts. In Section 3, the nine mappings from DST to PT are represented and the Cobb and Shenoy's requirements are investigated for these mappings. In Section 4, the invariance with respect to the marginalization property is descripted and the modified pignistic probability and the modified normalized plausibility transformation are proposed that are invariant under the marginalization process. In Section 5, these modified mapping are utilized in two new ambiguity measures and are used to compute the dependency of the variables of a social bliss problem to the person's happiness value. Finally, in Section 6, some concluding remarks are made.

2 Theoretical Background

In the probability theory, a PDF $p: \Omega_X \rightarrow [0,1]$ assigns values to $\Omega_X = \{x_1, x_2, \dots, x_n\}$ (the state space of variable X), where $p(x_i) \geq 0$, and $\sum_{i=1:n} p(x_i) = 1$.

Definition 1. If Ω_X and Ω_Y are the state spaces of variables X and Y, then the joint state space is denoted by $\Omega_{XY} = \Omega_X \times \Omega_Y$ and $p_{XY}: \Omega_{XY} \rightarrow [0,1]$ is the corresponding joint PDF.

Marginalization in the probability theory involves addition over the state space of the variables being eliminated. Suppose p_{XY} is a joint PDF for Ω_{XY}, the marginal PDF for Ω_X is $p_X(x) = \sum_Y p_{XY}(x, y)$. Combination in PT with Bayes' rule involves "point wise" multiplication of probability density functions. Suppose p_X is a PDF for Ω_X and p_Y is a PDF for Ω_Y. Then, p_{XY} is a PDF for Ω_{XY} and defined as follows: $p_{XY}(z) = (p_X \otimes p_Y)(z) = K^{-1} p_X(x) p_Y(y)$ for each $z \in \Omega_{XY}$, where K is the normalization constant.

Dempster-Shafer theory is an imprecise probability theory in which a basic probability assignment (BPA) assigns values to the subsets of the state space [19, 20]. The function $m: 2^{\Omega_X} \rightarrow [0,1]$ is a BPA on the power set of $\Omega_X = \{x_1, x_2, \dots, x_n\}$, where $m(\emptyset) = 0$, $m(A) \geq 0$, and $\sum_{A \in 2^{\Omega_X}} m(A) = 1$. Any element in 2^{Ω_X} with a non-zero BPA is called a focal element. Two other functions defined in DST are the belief function ($Bel(A) = \sum_{B \subseteq A} m(B)$) and the plausibility function ($Pl(A) = \sum_{A \cap B \neq \emptyset} m(B)$).

Topics such as joint state space, projection, and marginalization in DST are defined as follows:

Definition 2. If 2^{Ω_X} and 2^{Ω_Y} are the state spaces of variables X and Y with cardinalities $2^{|\Omega_X|}$ and $2^{|\Omega_Y|}$, then the joint state space is denoted by $2^{\Omega_{XY}}$ and defined by $\Omega_{XY} = \Omega_X \times \Omega_Y$. Its cardinality is $2^{|\Omega_{XY}|}$.

Example 1. If $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2, y_3\}$ are the state spaces of X and Y, then the joint state space in DST will be $2^{\Omega_{XY}}$ ($\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$), and have $2^{|\Omega_{XY}|} = 2^6 = 64$ members. To put it simply, we introduce the following notation: $\Omega_{XY} = \Omega_X \times \Omega_Y = \{Z_{11}, Z_{12}, Z_{13}, Z_{21}, Z_{22}, Z_{23}\}$.

If Ω_X and Ω_Y are the state spaces of two variables and $\Omega_{XY} = \Omega_X \times \Omega_Y$ is the corresponding joint state space, then the projection of any subset $A \subseteq \Omega_{XY}$ on Ω_X is denoted by $A^{\downarrow \Omega_X}$. This projection is shown in Fig. 1.

Definition 3. If $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ is a joint BPA on Ω_{XY}, then the marginal of m_{XY} on Ω_X is denoted by $m_{XY}^{\downarrow \Omega_X}$, and given by:

$$m_X(B) = m_{XY}^{\downarrow \Omega_X}(B) = \sum_{A \subseteq \Omega_{XY}, A \downarrow \Omega_X = B} m_{XY}(A) \quad \forall \subseteq \Omega_X \quad (1)$$

We will show that the number of *marginal singletons* in the joint state space is a major factor in the current study, as emphasized by the following definitions.

Definition 4. If $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ is a joint BPA on Ω_{XY}, then the marginal singletons of subset $A \in 2^{\Omega_{XY}}$ will be the all members of Ω_X and Ω_Y that exist in A.

Definition 5. If $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ is a joint BPA on Ω_{XY}, and if $x_i \in \Omega_X$ is a marginal singleton of $A \in 2^{\Omega_{XY}}$, then the number of marginal singletons x_i is denoted by $\sharp (x_i \in A)$.

To illustrate the point, the marginal singletons number has been calculated for the subset $A = \{Z_{12}, Z_{13}, Z_{23}\}$, which is a member of the joint state space in Example 1. The marginal singletons are x_1, x_2, y_2, and y_3.

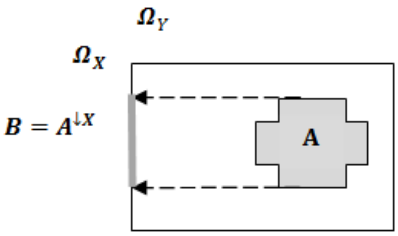

Fig. 1 Projection of subset $A \subseteq \Omega_{XY}$ on Ω_X.

There are two x_1, one x_2, one y_2, and two y_3 in subset A and we can write $\#(x_1 \in A) = 2$, $\#(x_2 \in A) = 1$, $\#(y_2 \in A) = 1$, and $\#(y_3 \in A) = 2$. Note that the number of marginal singletons of any member of a joint state space in PT is 1 (for instance, $(x_1, y_1) \in \Omega_{XY}$ has one x_1 and one y_1). However, the marginal singletons number of any subset of the joint state space in DST is not necessarily one.

Definition 6. Let $m_X: 2^{\Omega_X} \to [0,1]$ and $m_Y: 2^{\Omega_Y} \to [0,1]$ be two equally reliable and independence BPAs, the combination is calculated by Dempster's rule of combination in the following manner [21]:

$$m_{XY}(Z) = (m_X \oplus m_Y)(Z) = \frac{\sum_{X \cap Y = Z} m_X(X).m_Y(Y)}{1-K} \quad (2)$$

$\forall\ X \subseteq \Omega_X \&\ \ Y \subseteq \Omega_Y$

where, $K = \sum_{X \cap Y = \emptyset} m_X(X).m_Y(Y)$ represents the conflict.

3 DST to PT Transformations and Their Requirements

Several mappings have been proposed to extract probability distributions from a BPA. The mappings that are studied in the current paper, are defined as follows:

Definition 7. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X, then $BetP_m$ denotes the corresponding probability function obtained using the pignistic probability, which is defined for each singleton $x \in \Omega_X$ as follows [3, 22]:

$$BetP_m(\{x\}) = \sum_{\substack{A \subseteq \Omega_X \\ x \in A}} {m(A)}/{|A|} \quad (3)$$

Definition 8. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X and $P_{6.1}$ denotes the probability function used in the AU proposed by Klir, then, $P_{6.1}$ is obtained by the optimization process of the algorithm 6.1 [8].

Definition 9. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X and PlP_m denotes the corresponding probability function obtained using the normalized plausibility transformation, then $PlP_m$is defined for each singleton $x \in \Omega_X$ as follows [1]:

$$PlP_m(\{x\}) = \frac{1}{\Delta} \sum_{\substack{A \subseteq \Omega_X \\ x \in A}} m(A) = \frac{1}{\Delta} Pl(\{x\}) \quad (4)$$

where Δ is the normalization factor in which:

$$\sum_{x \in \Omega_X} PlP_m(\{x\}) = 1 \quad (5)$$

Definition 10. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X, then $PrPl_m$ and $PrBel_m$ are mathematically defined as follows for each singleton $x \in \Omega_X$ [1]:

$$PrPl_m(\{x\}) = Pl(\{x\}).\sum_{\substack{A \in 2^{\Omega_X} \\ x \in A}} \frac{1}{CS[Pl(A)]} m(A) \quad (6)$$

$$PrBel_m(\{x\}) = Bel(\{x\}).\sum_{\substack{A \in 2^{\Omega_X} \\ x \in A}} \frac{1}{CS[Bel(A)]} m(A) \quad (7)$$

where the denominators involved in the formulas are given by the compound to sum of singletons CS[.] operator defined in [23]:

$$CS[Pl(A)] \triangleq \sum_{\substack{A_i \in 2^{\Omega_X} \\ |A_i|=1 \\ \cup_i A_i = A}} Pl(A_i) \quad (8)$$

$$CS[Bel(A)] \triangleq \sum_{\substack{A_i \in 2^{\Omega_X} \\ |A_i|=1 \\ \cup_i A_i = A}} Bel(A_i) \quad (9)$$

Definition 11. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X, The mapping proportional to all plausibilities is defined as follows [1]:

$$PraPl_m(\{x\}) = Bel_m(\{x\}) + \epsilon.Pl_m(\{x\}) \quad (10)$$

and

$$\epsilon \triangleq \frac{1 - \sum_{B \in 2^{\Omega_X}} Bel(B)}{\sum_{B \in 2^{\Omega_X}} Pl(B)} \quad (11)$$

Definition 12. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X, The hybrid pignistic probability is defined as follows [1]:

$$PrHyb_m(\{x\}) = PraPl_m(\{x\}).\sum_{\substack{A \in 2^{\Omega_X} \\ x \in A}} \frac{1}{CS[PraPl_m(A)]} m(A) \quad (12)$$

$$CS[PraPl_m(A)] \triangleq \sum_{\substack{A_i \in 2^{\Omega_X} \\ |A_i|=1 \\ \cup_i A_i = A}} PraPl_m(A_i) \quad (13)$$

Definition 13. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X, $CuzzP_m$ is defined on any $x_i \in \Omega_X = \{x_1, x_2, \dots, x_n\}$ as follows [2]:

$$CuzzP_m(x_i) = m(x_i) + \frac{\Delta(x_i)}{\sum_{j=1}^{n} \Delta(x_i)} \times TNSM \quad (14)$$

where $\Delta(x_i) \triangleq Pl(x_i) - m(x_i)$ and TNSM is Total Non Specific Mass, and compute by:

$$TNSM = 1 - \sum_{j=1}^{n} m(x_j) = \sum_{A \in 2^{\Omega_X}, |A|>1} m(A) \quad (15)$$

Definition 14. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X, the generalized pignistic probability denoted $DSmP_\epsilon$ is defined by Smarandache *et al.* as follows [7]:

$$DSmP_\epsilon(A) = \sum_{B \in 2^{\Omega_X}} \frac{\sum_{\substack{Z \subseteq A \cap B \\ |Z|=1}} m(Z) + \epsilon.|A \cap B|}{\sum_{\substack{Z \subseteq B \\ |Z|=1}} m(Z) + \epsilon.|B|} m(B) \quad (16)$$

and ϵ is a tuning parameter.

Five necessary requirements of a probability transformation have been investigated for $PrPl_m$ and $BetP_m$ by Cobb and Shennoy [15]. These properties are: *Invariance with respect to marginalization, Invariance with respect to combination, Idempotency, Unique most probable state, and Non-Unique most probable state.* Now, we study these conditions for all above mappings with some examples.

Definition 15 (**P. 1**). If $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ is an arbitrary joint BPA on Ω_{XY}, m_X is the marginal BPA on Ω_X, and $T_{m_{XY}}$ and T_{m_X} are the probabilistic transformations of m_{XY} and m_X respectively, then T_m is invariant with respect to marginalization if and only if:

$$T_{m_X} = (T_{m_{XY}})^{\downarrow \Omega_X} \quad (17)$$

This means that marginal probability distribution obtained from the joint probability transformation is equal to the probability distribution obtained from the marginal BPAs. This concept is shown in Fig. 2. Now we want to know which mapping satisfies this property. For this purpose the Klir's example has been presented for the pignistic probability in [14], is used.

Example 2 [14]. Let $m: 2^{\Omega_{XY}} \rightarrow [0,1]$ be the joint BPA for $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2\}$ ($\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\} = \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}$). Which mapping is invariant with respect to the marginalization process?

$$\begin{cases} m(\{Z_{11}, Z_{12}, Z_{21}\}) = a \\ m(\{Z_{22}\}) = 1 - a \end{cases}, 0 \leq a \leq 1 \ . \quad (18)$$

The results are listed in the fifth column of the Table 1. In the first column of the table, the probability transformations are listed. The joint probability transformations are listed in the second column, and the probability distributions obtained from the marginal BPA's are listed in the columns 3 and 4. The results illustrate that none of the mappings satisfy this condition.

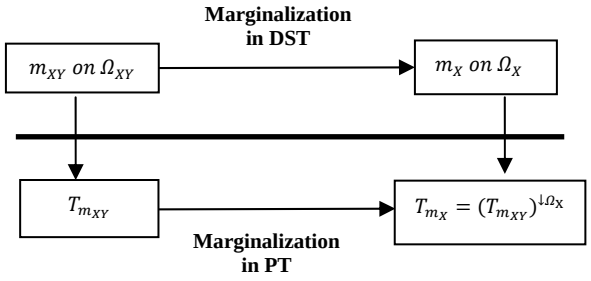

Fig. 2 Invariance with respect to the marginalization of a mapping.

Definition 16 (**P. 2**). Suppose $m_1, m_2, \ldots, m_k: 2^{\Omega_X} \rightarrow [0,1]$ be K independent BPAs on Ω_X, and $T_{m_1}, T_{m_2}, \ldots, T_{m_K}$ be the corresponding probability functions. If $m = m_1 \oplus m_2 \oplus \ldots \oplus m_k$ is the joint BPA, then,

$$T_{(m_1 \oplus m_2 \oplus \ldots \oplus m_k)} = T_{m_1} \otimes T_{m_2} \otimes \ldots \otimes T_{m_K} \quad (19)$$

It means that the probability transformation of a BPA obtained from the combination of K independent

Table 1 Invariance with respect to marginalization of the mappings.

T_m	$T_{m_{XY}}$	T_{m_X}	T_{m_Y}	*P. 1*
$P_{6.1}$	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{cases} \begin{bmatrix} a \\ 1-a \end{bmatrix}; & if a < 1/2 \\ \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}; & if a \geq 1/2 \end{cases}$	$\begin{cases} [a \quad 1-a]; & if a < 1/2 \\ [1/2 \quad 1/2]; & if a \geq 1/2 \end{cases}$	No
$BetP_m$	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{bmatrix} a/2 \\ 1-a/2 \end{bmatrix}$	$[a/2 \quad 1-a/2]$	No
PlP_m	$\begin{bmatrix} \frac{a}{1+2a} & \frac{a}{1+2a} \\ \frac{a}{1+2a} & \frac{1-a}{1+2a} \end{bmatrix}$	$\begin{bmatrix} \frac{a}{1+a} \\ \frac{1}{1+a} \end{bmatrix}$	$\begin{bmatrix} \frac{a}{1+a} & \frac{1}{1+a} \end{bmatrix}$	No
$PrPl_m$	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{bmatrix} \frac{a^2}{1+a} \\ \frac{1-a^2+a}{1+a} \end{bmatrix}$	$\begin{bmatrix} \frac{a^2}{1+a} & \frac{1-a^2+a}{1+a} \end{bmatrix}$	No
$PrBel_m$	$\begin{bmatrix} NaN & NaN \\ NaN & 1-a \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$[0 \quad 1]$	No
$PraPl_m$	$\begin{bmatrix} \frac{a^2}{1+2a} & \frac{a^2}{1+2a} \\ \frac{a^2}{1+2a} & \frac{1+2a-3a^2}{1+2a} \end{bmatrix}$	$\begin{bmatrix} \frac{a^2}{1+a} \\ \frac{1-a^2+a}{1+a} \end{bmatrix}$	$\begin{bmatrix} \frac{a^2}{1+a} & \frac{1-a^2+a}{1+a} \end{bmatrix}$	No
$PrHyb_m$	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{bmatrix} \frac{a^3}{1+a} \\ \frac{1-a^3+a}{1+a} \end{bmatrix}$	$\begin{bmatrix} \frac{a^3}{1+a} & \frac{1-a^3+a}{1+a} \end{bmatrix}$	No
$CuzzP_m$	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{bmatrix} a/2 \\ 1-a/2 \end{bmatrix}$	$[a/2 \quad 1-a/2]$	No
$DSmP_\epsilon$	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{bmatrix} \frac{\epsilon a}{1-a+2\epsilon} \\ \frac{1-a+2\epsilon-\epsilon a}{1-a+2\epsilon} \end{bmatrix}$	$\begin{bmatrix} \frac{\epsilon a}{1-a+2\epsilon} & \frac{1-a+2\epsilon-\epsilon a}{1-a+2\epsilon} \end{bmatrix}$	No

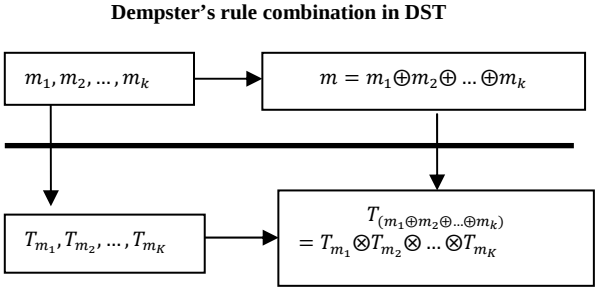

Fig. 3 Independency from Dempster's rule of combination of a mapping.

Table 2 Independency from Dempster's rule of combination of the mappings.

T_m	m_1	m_2	$m_{12} = m_1 \oplus m$	*P.* 2
$P_{6.1}$	[0.5 0.25 0.25]	[0.5 0.5 0]	[0.5 0.5 0]	No
$BetP_m$	[0.5 0.25 0.25]	[0.5 0.5 0]	[0.5 0.5 0]	No
PlP_m	[0.33 0.33 0.33]	[0.5 0.5 0]	[0.5 0.5 0]	Yes
$PrPl_m$	[0.5 0.25 0.25]	[0.5 0.5 0]	[0.5 0.5 0]	No
$PrBel_m$	[1 0 0]	[0.5 0.5 0]	[0.5 0.5 0]	No
$PraPl_m$	[0.66 0.167 0.167]	[0.5 0.5 0]	[0.5 0.5 0]	No
$PrHyb_m$	[0.66 0.167 0.167]	[0.5 0.5 0]	[0.5 0.5 0]	No
$CuzzP_m$	[0.5 0.25 0.25]	[0.5 0.5 0]	[0.5 0.5 0]	No
$DSmP_\epsilon$	[0.5 0.25 0.25]	[0.5 0.5 0]	[0.5 0.5 0]	No

Table 3 Idempotency property of the mappings.

T_m	x_1	x_2	x_3	*P. 3*
$P_{6.1}$	0.5	0.25	0.25	No
$BetP_m$	0.5	0.25	0.25	No
PlP_m	1/3	1/3	1/3	Yes
$PrPl_m$	0.5	0.25	0.25	No
$PrBel_m$	0.5	NaN	NaN	No
$PraPl_m$	4/6	1/6	1/6	No
$PrHyb_m$	0.5	0.25	0.25	No
$CuzzP_m$	0.5	0.25	0.25	No
$DSmP_\epsilon$	0.5	0.25	0.25	No

BPAs with the Dempster's rule of combination is equal to the combination of *K* probability transformations of *K* BPAs with Bayes' rule in PT. This concept is shown in Fig. 3. Similar to Example 2, this condition is investigated for the above mappings by another example.

Example 3. Suppose $m_1 \& m_2 : 2^{\Omega_X} \to [0,1]$ are two BPAs on $\Omega_X = \{x_1, x_2, x_3\}$, $m_1(\{x_1\}) = 0.5$, $m_1(\{x_2, x_3\}) = 0.5$ and $m_2(\{x_1, x_2\}) = 1$. Then the combination of this BPAs is: $m_{12}(\{x_1\}) = 0.5$, and $m_{12}(\{x_2\}) = 0.5$. The independency from combination property is shown in the Table 2. The results show that just the normalized plausibility transformation satisfies this condition. The proof of this claim follows directly from the proof of the Proposition 2 in [23].

Definition 17 (**P. 3**). Suppose $m: 2^{\Omega_X} \to [0,1]$ be the BPA on Ω_X, and T_m be the corresponding probability function. If m is idempotent with respect to the Dempster's rule, i.e., $= m \oplus m$, then T_m is idempotent with respect to the Bayes' rule, i.e., $T_m = T_m \otimes T_m$.

This is a corollary of P. 2. In Example 4, the idempotency property of the above mappings is illustrated.

Example 4. Let $m: 2^{\Omega_X} \to [0,1]$ be a BPA on $\Omega_X = \{x_1, x_2, x_3\}$ and $m(\{x_1\}) = 0.5$ and $m(\{x_2, x_3\}) = 0.5$ .Check the idempotency property for the above probability transformations. The result is shown in Table 3. It is understood that just the normalized plausibility transformation satisfies this condition.

Definition 18 (**P. 4**). Suppose $m: 2^{\Omega_X} \to [0,1]$ be the BPA on Ω_X, $m^n = m_1 \oplus m_2 \oplus \dots \oplus m_n$ be the n times combination of m, and $lim_{n\to\infty} m^n(x) = 1$. Let T_m denotes the probability transformation corresponding to the m, $(T_m)^n = T_{m_1} \otimes T_{m_2} \otimes \dots \otimes T_{m_n}$ be the n times combination of T_m, and ${T_m}^\infty$ denotes $\lim_{n\to\infty}(T_m)^n$. Then, ${T_m}^\infty(x) = 1$, and ${T_m}^\infty(y) = 0$ for all $y \in \Omega_X \backslash \{x\}$.

It means that if a unique most probable state x exists in the BPA m, then the corresponding probability function should have x as its most probable state. From Example 5, it can be found that this property is satisfied just by the normalized plausibility transformation.

Example 5. If $m: 2^{\Omega_X} \to [0,1]$ be a BPA on $\Omega_X = \{x_1, x_2, x_3, x_4, x_5\}$, $m(\{x_1\}) = 0.45$, $m(\{x_2\}) = 0.15$, $m(\{x_3\}) = 0.05$, and $m(\{x_2, x_3, x_4, x_5\}) = 0.35$. Then we have, $m^\infty(\{x_2\}) = 1$. After the computation of the mappings, we have: $Pl\_Pm^\infty(\{x_2\}) = 1$, $PrBel^\infty(\{x_1\}) = 1$, $PrHyb^\infty(\{x_1\}) = 1$, $P6.1^\infty(\{x_1\}) = 1$, $BetPm^\infty(\{x_1\}) = 1$, $PrPl^\infty(\{x_1\}) = 1$, $PraPl^\infty(\{x_1\}) = 1$, $CuuzP^\infty(\{x_1\}) = 1$, $DSmP^\infty(\{x_1\}) = 1$. So the normalized plausibility transformation has maximum unique state.

Definition 19 (**P. 5**). Suppose $m: 2^{\Omega_X} \to [0,1]$ is the BPA on Ω_X such that $lim_{n\to\infty} m^n(x) = lim_{n\to\infty} m^n(y)$ for all $x, y \in A \subseteq \Omega_X$ and $lim_{n\to\infty} m^n(z) = 0$ for all $z \in \Omega_X \backslash A$. Let T_m denotes the probability transformation of m, and ${T_m}^\infty$ denotes $lim_{n\to\infty}(T_m)^n$. Then ${T_m}^\infty(x) = {T_m}^\infty(y) = \frac{1}{|A|}$ for all $x, y \in A$, and ${T_m}^\infty(z) = 0$ for all $z \in \Omega_X \backslash A$.

Example 6. If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on $\Omega_X = \{x_1, x_2, x_3, x_4, x_5\}$, $m(\{x_1\}) = 0.4$, $m(\{x_2\}) = 0.15$, $m(\{x_3\}) = 0.15$, and $m(\{x_2, x_3, x_4, x_5\}) = 0.3$. Then we have, $m^\infty(\{x_2\}) = m^\infty(\{x_3\}) = 0.5$. The non-unique states are computed as follows: $Pl\_Pm^\infty(\{x_2\}) = Pl\_Pm^\infty(\{x_3\}) = 0.5$, $PrBel^\infty(\{x_1\}) = 1$, $PrHyb^\infty(\{x_1\}) = 1$, $P6.1^\infty(\{x_1\}) = 1$, $BetPm^\infty(\{x_1\}) = 1$, $PrPl^\infty(\{x_1\}) = 1$, $PraPl^\infty(\{x_1\}) = 1$, $CuuzP^\infty(\{x_1\}) = 1$ and $DSmP^\infty(\{x_1\}) = 1$. So only the normalized plausibility transformation satisfies this condition.

Table 4 Survey of the mappings and their properties.

T_m	*P. 1*	*P. 2*	*P. 3*	*P. 4*	*P. 5*
$P_{6.1}$	NO	NO	NO	NO	NO
$BetP_m$	NO	NO	NO	NO	NO
PlP_m	YES	YES	YES	YES	NO
$PrPl_m$	NO	NO	NO	NO	NO
$PrBel_m$	NO	NO	NO	NO	NO
$PraPl_m$	NO	NO	NO	NO	NO
$PrHyb_m$	NO	NO	NO	NO	NO
$CuzzP_m$	NO	NO	NO	NO	NO
$DSmP_\epsilon$	NO	NO	NO	NO	NO

The results are summarized in Table 4. As a consequence two important results are extracted from the Table 4.

1. The independency from the Dempster's rule of combination of a probability transformation will also involve the last three conditions. So two main requirements for a conversion from the DST to the PT are *invariance with respect to the marginalization* and *invariance with respect to the Dempster's rule of combination.*
2. According to the Table 4, it can be seen that only the normalized plausibility transformation satisfies four of the five conditions. In other words, if we want to choose the most justifiable mapping through the mentioned transformations, the normalized plausibility transformation is the best choice. Also the *invariance with respect to marginalization* problem of this mapping will be remained. This issue will be proven in the next sections.

4 Solving the Problem of the Invariance with Respect to the Marginalization Process

Invariance with respect to the marginalization process means that the marginal probability distribution of the joint probability transformation is equal to the probability distribution of the marginal BPAs. In this subsection, we will examine the reasons for the dependency of the pignistic probability and the normalized plausibility transformation on the marginalization process. To this end, we need to focus on the projection method in DST. The classic projection process in DST loses some probabilistic information, as shown in Example 7.

Example 7. If $m_1, m_2, m_3, m_4: 2^{\Omega_{XY}} \to [0,1]$ are four different joint BPAs with $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2, y_3\}$.

$$\begin{cases} m_1(\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{21}, \mathrm{Z}_{22}, \mathrm{Z}_{23}) = a \\ m_1(\mathrm{Z}_{23}) = 1 - a \end{cases},$$
$$\begin{cases} m_2(\mathrm{Z}_{11}, \mathrm{Z}_{21}, \mathrm{Z}_{13}, \mathrm{Z}_{23}) = a \\ m_2(\mathrm{Z}_{23}) = 1 - a \end{cases}$$
$$\begin{cases} m_3(\mathrm{Z}_{12}, \mathrm{Z}_{22}) = a \\ m_3(\mathrm{Z}_{23}) = 1 - a \end{cases}, \begin{cases} m_4(\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{23}) = a \\ m_4(\mathrm{Z}_{23}) = 1 - a \end{cases} \quad (20)$$

Then, there are different BPAs with different focal elements and we have,

$$\begin{aligned} &\{\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{21}, \mathrm{Z}_{22}, \mathrm{Z}_{23}\}^{\downarrow\Omega_X} \\ &= \{\mathrm{Z}_{11}, \mathrm{Z}_{21}, \mathrm{Z}_{13}, \mathrm{Z}_{23}\}^{\downarrow\Omega_X} = \{\mathrm{Z}_{12}, \mathrm{Z}_{22}\}^{\downarrow\Omega_X} \\ &= \{\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{23}\}^{\downarrow\Omega_X} = \{x_1, x_2\} \end{aligned} \quad (21)$$

Therefore, the projections of different subsets with different numbers of marginal singletons (three x_1 and three x_2 for the first subset, two x_1 and two x_2 for the second subset, one x_1 and one x_2 for the third subset, and three x_1 and one x_2 for the fourth subset) are equal. The four marginal BPAs on Ω_X are thus equal and can be given by:

$$m_X(\{x_2\}) = m_{XY}^{\downarrow\Omega_X}(\{x_2\}) = \sum_{A \subseteq \Omega_{XY}, A\downarrow\Omega_X = \{x_2\}} m_{XY}(A) = 1 - a \quad (22)$$

$$m_X(\{x_1, x_2\}) = m_{XY}^{\downarrow\Omega_X}(\{x_1, x_2\}) = \sum_{A \subseteq \Omega_{XY}, A\downarrow\Omega_X = \{x_1, x_2\}} m_{XY}(A) = a \quad (23)$$

In this example, there are four joint BPAs with different $BetP_{m_{XY}}$s, although their marginal pignistic probabilities computed from the marginal BPAs are equal. Additionally, there are four joint BPAs with different PlP_ms, although their marginal normalized plausibility transformations computed from the marginal BPAs are equal.

In other words, in the standard projection process, the number of marginal singletons (x_1 and x_2) that exists in the joint state space is not taken into account. This point explains why $BetP_m$ and PlP_m are not invariant under the marginalization process. To solve this problem, we need to consider the number of marginal singletons in the projection process. Therefore, we try to retain this information by defining the Projection Set and rewriting the marginalization formula as follows:

Definition 20. If $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ is the joint BPA on Ω_{XY}, then the Projection Set of Ω_{XY} on $B \subseteq \Omega_X$, is shown by $PrSt_{\downarrow B}$, which is the set of all joint state space members such that $A^{\downarrow\Omega_X} = B$ and is given by:

$$PrSt_{\downarrow B} = \{A | A \subseteq \Omega_{XY}, A \downarrow \Omega_X = B\} \quad (24)$$

Definition 21. If $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ is the joint BPA defined on Ω_{XY}, then the marginal of m_{XY} over Ω_X based on $PrSt$ is denoted by $\acute{m}_{XY}^{\downarrow\Omega_X}$, and is computed as follows:

$$m'_{\Omega_{XY}}{}^{\downarrow\Omega_X}(B) = \sum_{C \in PrSt_{\downarrow B}} m_{XY}(C) \quad \forall B \subseteq \Omega_X \quad (25)$$

It should be noted that the results of the new marginalization procedure are almost identical with the classical method of the marginalization in Dempster-Shafer theory, only the formula has been little changed. Based on these changes, the pignistic probability and the normalized plausibility transformation could be modified as follows:

Definition 22. If $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ is a joint BPA defined over Ω_{XY}, then the modified pignistic probability is defined for each singleton $Z_i \in \Omega_{XY}$ and $x_i \in \Omega_X$ as follows:

$$B\acute{e}tP_{m_X}(x_i) = \sum_{B\subseteq\Omega_X, x_i\in B} \sum_{A\in PrSt_{\downarrow B}} \frac{m^{\Omega_{XY}}(A).\sharp(x_i \in A)}{|A|} \quad \forall x_i \in \Omega_X \tag{26}$$

$$B\acute{e}tP_{m_{XY}}\big((x_i, y_j)\big) = \sum_{\substack{A\subseteq\Omega_{XY} \\ (x_i,y_j)\in A}} \frac{m^{\Omega_{XY}}(A)}{|A|} \quad \forall (x_i, y_j) \in \Omega_{XY} \tag{27}$$

where $\sharp(x_i \in A)$ is the number of x_i in the subset A and $|A|$ denotes the cardinality of A.

Definition 23. If $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ is a joint BPA on Ω_{XY}., then the modified normalized plausibility transformation is defined for each singleton $Z_i \in \Omega_{XY}$ and $x_i \in \Omega_X$ as follows:

$$P\acute{l}P_{m_X}(x_i) = \frac{1}{\Delta} \sum_{B\subseteq\Omega_X, x_i\in B} \sum_{A\in PrSt_{\downarrow B}} m^{\Omega_{XY}}(A).\sharp(x_i \in A) \quad \forall x_i \in \Omega_X \tag{28}$$

$$P\acute{l}P_{m_{XY}}\big((x_i, y_j)\big) = \frac{1}{\Delta} \sum_{\substack{A\subseteq\Omega_{XY} \\ (x_i,y_j)\in A}} m^{\Omega_{XY}}(A) \quad \forall (x_i, y_j) \in \Omega_{XY} \tag{29}$$

where, Δ is the normalization factor.

Corollary 1 In one-dimensional state space,

1- The modified pignistic probability is reduced to the pignistic probability, i.e., $B\acute{e}tP_{m_X} = BetP_{m_X}$.

2- The normalized plausibility transformation and the modified normalized plausibility transformation are equal ($P\acute{l}P_{m_X} = PlP_{m_X}$).

Proof: In one-dimensional space we have,

$$m^{\Omega_{XY}}(A) = m_X(A) \implies \forall A \subseteq \Omega_X: \sharp(x_i \in A) = 1 \text{ and } PrSt_{\downarrow B} = B \tag{30}$$

Then,

$$B\acute{e}tP_{m_X}(x_i) = \sum_{B\subseteq\Omega_X, x_i\in B} \sum_{A\in PrSt_{\downarrow B}} \frac{m^{\Omega_{XY}}(A).\sharp(x_i \in A)}{|A|} = \sum_{B\subseteq\Omega_X, x_i\in B} \frac{m_X(B)}{|B|} = BetP_{m_X}(x_i) \quad \forall x_i \in \Omega_X \tag{31}$$

and,

$$P\acute{l}P_{m_X}(x_i) = \frac{1}{\Delta} \sum_{B\subseteq\Omega_X, x_i\in B} \sum_{A\in PrSt_{\downarrow B}} m^{\Omega_{XY}}(A).\sharp(x_i \in A) = \frac{1}{\Delta} \sum_{B\subseteq\Omega_X, x_i\in B} m_X(B) = PlP_{m_X}(x_i) \quad \forall x_i \in \Omega_X \tag{32}$$

Here, the invariance with respect to marginalization of the modified pignistic probability is expressed in the following proposition and its proof is given in Appendix A.

Proposition 1. Let $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ be a joint BPA over Ω_{XY}, $B\acute{e}tP_{m_{XY}}$ be the joint modified pignistic probability and $B\acute{e}tP_{m_X}$ be the modified pignistic probability of marginal m_X, then we have:

$$B\acute{e}tP_{m_X} = (B\acute{e}tP_{m_{XY}})^{\downarrow\Omega_X} \tag{33}$$

Proof: See Appendix A.

Now, invariance with respect to marginalization of $P\acute{l}P_{m_X}$ is expressed with the following proposition:

Proposition 2. Let $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ be a joint BPA on Ω_{XY}, $P\acute{l}P_{m_{XY}}$ be the joint modified normalized plausibility transformation and $P\acute{l}P_{m_X}$ be the modified normalized plausibility transformation of marginal m_X, then we have:

$$P\acute{l}P_{m_X} = (P\acute{l}P_{m_{XY}})^{\downarrow\Omega_X} \tag{34}$$

Proof: See Appendix B.

To clarify the point, the modified pignistic probability and the modified normalized plausibility transformation are computed for the BPAs of Example 2.

First, the joint probabilities are computed as follows:

$$B\acute{e}tP_{m_{XY}} = BetP_{m_{XY}} = \begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix} \tag{35}$$

$$PlP_{m_{XY}}(X,Y) = P\acute{l}P_{m_{XY}}(X,Y) = \begin{bmatrix} \frac{a}{1+2a} & \frac{a}{1+2a} \\ \frac{a}{1+2a} & \frac{1-a}{1+2a} \end{bmatrix} \tag{36}$$

As it can be seen in the Table 5, both the modified pignistic probability and the modified normalized plausibility transformation are invariant under the marginalization process ($B\acute{e}tP_{m_X} = (B\acute{e}tP_{m_{XY}})^{\downarrow\Omega_X}$, $B\acute{e}tP_{m_Y} = (B\acute{e}tP_{m_{XY}})^{\downarrow\Omega_Y}$, $P\acute{l}P_{m_X} = (P\acute{l}P_{m_{XY}})^{\downarrow\Omega_X}$ and $P\acute{l}P_{m_Y} = (P\acute{l}P_{m_{XY}})^{\downarrow\Omega_Y}$).

Example 8. If $m: 2^{\Omega_{XY}} \to [0,1]$ is a joint BPA on $\Omega_X = \{x_1, x_2, x_3\}$ and $\Omega_Y = \{y_1, y_2\}$, then we need to check the invariance with respect to marginalization concept for $BetP_m$, $B\acute{e}tP_m$, PlP_m and $P\acute{l}P_m$.

The joint state space is: $\Omega_{XY} = \Omega_X \times \Omega_Y = \{Z_{11}, Z_{12}, Z_{21}, Z_{22}, Z_{31}, Z_{32}\}$.

$$\begin{cases} m(\{Z_{11}, Z_{12}, Z_{21}, Z_{31}, Z_{32}\}) = \frac{a}{4} \\ m(\{Z_{31}, Z_{32}\}) = \frac{a}{4} \\ m(\{Z_{11}, Z_{12}, Z_{21}, Z_{31}\}) = \frac{a}{4} \\ m(\{Z_{22}\}) = \frac{a}{4} \\ m(\{Z_{21}, Z_{31}, Z_{32}\}) = 1-a \quad ,0 \le a \le 1 \end{cases} \tag{37}$$

Table 5 Projection Sets of X and Y, corresponding modified pignistic probabilities and modified normalized plausibility transformation.

2^{Ω_X}	$PrSt_{\downarrow(.)}$	$B\acute{e}tP_{\mathrm{m_X}}$	$P\hat{l}P_{m_X}$	2^{Ω_Y}	$PrSt_{\downarrow(.)}$	$B\acute{e}tP_{m_Y}$	$P\hat{l}P_{m_Y}$
x_1	$\{\emptyset\}$	$2a/3$	$\frac{2a}{1+2a}$	y_1	$\{\emptyset\}$	$2a/3$	$\frac{2a}{1+2a}$
x_2	$\{\{Z_{22}\}\}$	$1-2a/3$	$\frac{1}{1+2a}$	y_2	$\{\{Z_{22}\}\}$	$1-2a/3$	$\frac{1}{1+2a}$
x_1x_2	$\{\{Z_{11},Z_{12},Z_{21}\}\}$	…	…	y_1y_2	$\{\{Z_{11},Z_{12},Z_{21}\}\}$	…	…

First, we compute the joint probabilities as follows:

$$B\acute{e}tP_m(X,Y)=BetP_m(X,Y)=\begin{bmatrix}\frac{27a}{240} & \frac{27a}{240}\\ \frac{80-53a}{240} & \frac{60a}{240}\\ \frac{80-23a}{240} & \frac{80-38a}{240}\end{bmatrix} \quad (38)$$

$$P\hat{l}P_{m_{XY}}(X,Y)=PlP_{m_{XY}}(X,Y)=\begin{bmatrix}\frac{2a}{12} & \frac{2a}{12}\\ \frac{4-2a}{12} & \frac{a}{12}\\ \frac{4-a}{12} & \frac{4-2a}{12}\end{bmatrix} \quad (39)$$

Then, marginal BPAs of X, $BetP_{m_X}$, $B\acute{e}tP_{m_X}$, PlP_{m_X}, and $P\hat{l}P_{m_X}$ are calculated (Table 6). In the next step, the marginal BPA of Y, $BetP_{m_Y}$,$B\acute{e}tP_{m_Y}$,PlP_{m_Y}, and $P\hat{l}P_{m_Y}$ are computed (Table 7). Finally, a comparison will be made between the results of Table 6 and Table 7, suggesting that:

$$BetP_{m_X}(X)\neq\sum_Y BetP_m(X,Y) \quad (40)$$

$$BetP_{m_Y}(Y)\neq\sum_X BetP_m(X,Y) \quad (41)$$

But,

$$B\acute{e}tP_{m_X}(X)=\sum_Y B\acute{e}tP_m(X,Y) \quad (42)$$

$$B\acute{e}tP_{m_Y}(Y)=\sum_X B\acute{e}tP_m(X,Y) \quad (43)$$

and,

$$PlP_{m_X}(X)\neq\sum_Y PlP_{m_{XY}}(X,Y) \quad (44)$$

$$PlP_{m_Y}(Y)\neq\sum_X PlP_{m_{XY}}(X,Y) \quad (45)$$

but,

$$P\hat{l}P_{m_X}(X)=\sum_Y P\hat{l}P_{m_{XY}}(X,Y) \quad (46)$$

$$P\hat{l}P_{m_Y}(Y)=\sum_X P\hat{l}P_{m_{XY}}(X,Y) \quad (47)$$

Table 6 Marginal X,the Projection Set of X, $BetP_{m_X}$, $B\acute{e}tP_{m_X}$, PlP_{m_X}, and $P\hat{l}P_{m_X}$.

2^{Ω_X}	$m_X(.)$	$BetP_{m_X}$	PlP_{m_X}	$PrSt_{\downarrow(.)}$	$B\acute{e}tP_{m_X}$	$P\hat{l}P_{m_X}$
x_1	0	$\frac{2a}{12}$	$\frac{2a}{8}$	$\{\emptyset\}$	$\frac{54a}{240}$	$\frac{4a}{12}$
x_2	$\frac{a}{4}$	$\frac{6-a}{12}$	$\frac{4-a}{8}$	$\{\{Z_{22}\}\}$	$\frac{80+7a}{240}$	$\frac{4-a}{12}$
x_3	$\frac{a}{4}$	$\frac{6-a}{12}$	$\frac{4-a}{8}$	$\{\{Z_{31},Z_{32}\}\}$	$\frac{160-61a}{240}$	$\frac{8-3a}{12}$
x_1x_2	0	----	----	$\{\emptyset\}$	----	----
x_1x_3	0	----	----	$\{\emptyset\}$	----	----
x_2x_3	$1-a$	----	----	$\{\{Z_{21},Z_{31},Z_{32}\}\}$	----	----
$x_1x_2x_3$	$\frac{2a}{4}$	----	----	$\begin{Bmatrix}\{Z_{11},Z_{12},Z_{21},Z_{31},Z_{32}\}\\ \{Z_{11},Z_{12},Z_{21},Z_{31}\}\end{Bmatrix}$	----	----

Table 7 Marginal Y, the Projection Set of Y, $BetP_{m_Y}$, $B\acute{e}tP_{m_Y}$,PlP_{m_Y}, and $P\hat{l}P_{m_Y}$.

2^{Ω_Y}	$m_Y(.)$	$BetP_{m_Y}$	PlP_{m_Y}	$PrSt_{\downarrow(.)}$	$B\acute{e}tP_{m_Y}$	$P\hat{l}P_{m_Y}$
y_1	0	$\frac{4-a}{8}$	$\frac{4-a}{8-a}$	$\{\emptyset\}$	$\frac{160-49a}{240}$	$\frac{8-a}{12}$
y_2	$\frac{a}{4}$	$\frac{4+a}{8}$	$\frac{4}{8-a}$	$\{\{Z_{22}\}\}$	$\frac{80+49a}{240}$	$\frac{4+a}{12}$
y_1y_2	$1-\frac{a}{4}$	----	----	$\begin{Bmatrix}\{Z_{11},Z_{12},Z_{21},Z_{31},Z_{32}\},\{Z_{31},Z_{32}\}\\ \{Z_{11},Z_{12},Z_{21},Z_{31}\},\{Z_{21},Z_{31},Z_{32}\}\end{Bmatrix}$	----	----

5 Applications of the Proposed Mappings

Computing the amount of uncertainty or information contained in an event is of crucial importance in many applications in decision-making systems. To calculate the amount of uncertainty, we need to define a measure. Shannon entropy (H(x)) is an uncertainty measure in PT proposed by Shannon [24]. The different types of uncertainty proposed in various theories have been classified by Klir and Yuan in [25]. Dempster-Shafer Theory is an extension of the probability theory and the set theory, and as such, it is able to represent two types of uncertainty, i.e., nonspesifity and discord. Klir proposed AU as an aggregated uncertainty measure that computes nonspecificity and discord simultaneously [9]. He posited that any aggregate uncertainty measure such as AU must satisfy five requirements including *Probability consistency*, *Set consistency*, *Range*, *Subadditivity* and *Additivity*. Jousselme *et al.* proposed another aggregated uncertainty measure called AM based on the pignistic probability [13]. They proved that AM satisfies the five requirements of an aggregate uncertainty measure. But, Klir and Lewis showed that the proof of AM subadditivity provided by Jousselme *et al.* was wrong [14]. They referred to the dependency of the pignistic probability on the marginalization process to support their argument.

Similar to the pignistic probability that is used in AM, we can exploit the other DST to PT transformations to measure the amounts of ambiguity in DST. But the Table 3 indicates that the all mapping are dependent to the marginalization process and so the corresponding ambiguity measures are not subadditive. In Section 4, we proposed $\boldsymbol{Bet\acute{P}_m}$ and $\boldsymbol{Pl\acute{P}_m}$ that are invariant under the marginalization process and so are adequate to use in the ambiguity measure. Therefore the ambiguity measures based on $\boldsymbol{Bet\acute{P}_m}$ and $\boldsymbol{Pl\acute{P}_m}$ will be subadditive.

Now, similar to the entropy measure in PT, we have two new aggregate uncertainty measures in DST for computing the amounts of ambiguity. The question is where can be used these ambiguity measures. We attempted to use these measures for computing the amounts of dependency between two variables. As we know, mutual information (MI) as a tool for measuring the dependency between two variables is used in many applications in probability theory [26]. Similar to the mutual information in probability theory, the mutual ambiguity based on $\boldsymbol{Bet\acute{P}_m}$ and $\boldsymbol{Pl\acute{P}_m}$ can be used for computing the dependency between two variables in DST.

Shahpari *et al.* in [27], used the mutual ambiguity measure based on $\boldsymbol{Bet\acute{P}_m}$ called $\boldsymbol{MAM}$, in a threat assessment problem constructed by a Dempster-Shafer network. In their paper, *MAM* is used for computing the influence of the network input variables to the threat value.

In the similar way, we introduce the ambiguity measure and the mutual ambiguity measure based on $\boldsymbol{Bet\acute{P}_m}$ and $\boldsymbol{Pl\acute{P}_m}$ as follows:

Definition 24. If $m: 2^{\Omega_X} \rightarrow [0,1]$ is a BPA on Ω_X and $Bet\acute{P}_m$ and $Pl\acute{P}_m$ are DST to PT transformations, then the corresponding ambiguity measures are given by:

$$MAM(m) = -\sum_{x\in\Omega_X} Bet\acute{P}_m(x).\log_2\left(Bet\acute{P}_m(x)\right) \tag{48}$$

$$AM^{Pl\acute{P}_m}(m) = -\sum_{x\in\Omega_X} Pl\acute{P}_m(x).\log_2\left(Pl\acute{P}_m(x)\right) \tag{49}$$

Definition 25. If $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ is an arbitrary joint BPA on Ω_{XY}, the associated marginal BPAs are m_X and m_Y, then mutual ambiguity measures based on $Bet\acute{P}_m$ and $Pl\acute{P}_m$, are given by:

$$MAM(X;Y) = MAM(X) + MAM(Y) - MAM(X,Y) \tag{50}$$

$$AM^{Pl\acute{P}_m}(X;Y) = AM^{Pl\acute{P}_m}(X) + AM^{Pl\acute{P}_m}(Y) - AM^{Pl\acute{P}_m}(X,Y) \tag{51}$$

Example 9. Let us consider the issue of the social bliss and the factors that affect a person's happiness. Suppose that there are five independent parameters such as social acceptability (SA), hope for the future (HF), poverty (P), feeling of security (FS), and fulfillment of emotional needs (FE). The relationships between these factors and the target variable, social bliss (SB), are modeled by the expert knowledge expressed by some rules. Then according to the implication rule in [28-29], each of the rules can be represented by a BPA.

For example, an expert explains his opinion about the effect of social acceptability on the social bliss in the following two rules: 1) if the person has a good level of acceptability, then with certainty between 0.5 to 0.8 he feel happiness; and 2) if the person has no social acceptability, then with certainty between 0.3 to 0.6 he does not feel happiness. To model these rules, suppose that the state space of SA is $\Omega_{SA} = \{sa = 0, sa = 1\}$ and the state space of SB is $\Omega_{SB} = \{sb = 0, sb = 1\}$. Now, These rules are rewritten as: "(SA=1)→(SB=1) with confidence between 0.5 to 0.8." and "(SA=0)→(SB=0) with confidence between 0.3 to 0.6." Then, according to the implication rule in [28] the joint BPA is computed as follows:

The joint state space will be the power set of $\Omega_{SA,SB} = \Omega_{SA} \times \Omega_{SB} = \{(0,0),(0,1),(1,0),(1,1)\} = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and we have,

$$\begin{aligned} &m_{SA,SB}(\{Z_{00}, Z_{10}\}) = 0.06 \\ &m_{SA,SB}(\{Z_{01}, Z_{10}\}) = 0.08 \\ &m_{SA,SB}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.06 \\ &m_{SA,SB}(\{Z_{00}, Z_{11}\}) = 0.15 \end{aligned} \tag{52}$$

$m_{SA,SB}(\{Z_{01}, Z_{11}\}) = 0.2$
$m_{SA,SB}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.15$
$m_{SA,SB}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.09$
$m_{SA,SB}(\{Z_{01}, Z_{10}, Z_{11}\}) = 0.12$
$m_{SA,SB}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.09$

Similar to the above modeling, the state space of HF is $\Omega_{HF} = \{hf = 0, hf = 1\}$ and the expert rules and the joint BPAs are given as follows:

(SA=1)→(SB=1) with confidence between 0.5 to 0.8.

(SA=0)→(SB=0) with confidence between 0.3 to 0.6.

$m_{HF,SB}(\{Z_{00}, Z_{10}\}) = 0.05$
$m_{HF,SB}(\{Z_{01}, Z_{10}\}) = 0.02$
$m_{HF,SB}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.03$
$m_{HF,SB}(\{Z_{00}, Z_{11}\}) = 0.4$
$m_{HF,SB}(\{Z_{01}, Z_{11}\}) = 0.16$ (53)
$m_{HF,SB}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.24$
$m_{HF,SB}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.05$
$m_{HF,SB}(\{Z_{01}, Z_{10}, Z_{11}\}) = 0.02$
$m_{HF,SB}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.03$

For FE with the state space $\Omega_{FE} = \{fe = 0, fe = 1\}$ we have,

(FE=1)→(SB=1) with confidence between 0.6 to 0.7.

(FE=0)→(SB=0) with confidence between 0.2 to 0.5.

$m_{FE,SB}(\{Z_{00}, Z_{10}\}) = 0.02$
$m_{FE,SB}(\{Z_{01}, Z_{10}\}) = 0.03$
$m_{FE,SB}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.05$
$m_{FE,SB}(\{Z_{00}, Z_{11}\}) = 0.12$
$m_{FE,SB}(\{Z_{01}, Z_{11}\}) = 0.18$ (54)
$m_{FE,SB}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.3$
$m_{FE,SB}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.06$
$m_{FE,SB}(\{Z_{01}, Z_{10}, Z_{11}\}) = 0.09$
$m_{FE,SB}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.15$

For FS with the state space $\Omega_{FS} = \{fs = 0, fs = 1\}$, the expert knowledge and the joint BPA are given by:

(FS=1)→(SB=1) with confidence between 0.2 to 0.5.

(FS=0)→(SB=0) with confidence between 0.9 to 0.98.

$m_{FS,SB}(\{Z_{00}, Z_{10}\}) = 0.27$
$m_{FS,SB}(\{Z_{01}, Z_{10}\}) = 0.024$
$m_{FS,SB}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.006$
$m_{FS,SB}(\{Z_{00}, Z_{11}\}) = 0.18$
$m_{FS,SB}(\{Z_{01}, Z_{11}\}) = 0.016$ (55)
$m_{FS,SB}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.004$
$m_{FS,SB}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.45$
$m_{FS,SB}(\{Z_{01}, Z_{10}, Z_{11}\}) = 0.04$
$m_{FS,SB}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.01$

Finally, for P with the state space $\Omega_P = \{p = 0, p = 1\}$ we have,

(P=0)→(SB=1) with confidence between 0.6 to 0.8.

(P=1)→(SB=0) with confidence between 0.7 to 0.9.

$m_{P,SB}(\{Z_{00}, Z_{10}\}) = 0.12$
$m_{P,SB}(\{Z_{01}, Z_{10}\}) = 0.42$
$m_{P,SB}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.06$
$m_{P,SB}(\{Z_{00}, Z_{11}\}) = 0.04$
$m_{P,SB}(\{Z_{01}, Z_{11}\}) = 0.14$ (56)
$m_{P,SB}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.02$
$m_{P,SB}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.04$
$m_{P,SB}(\{Z_{01}, Z_{10}, Z_{11}\}) = 0.14$
$m_{P,SB}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.02$

Now, we want to identify which variables of the problem are more influential on the social bliss. To this end, $\boldsymbol{MAM(X;Y)}$ and $\boldsymbol{AM^{Pl\dot{P}_m}(X;Y)}$ are employed to compute the dependency of the paired variables (SA,SB), (HF,SB), (FE,SB), (FS,SB), and (P,SB). From Table 8 it can be observed that HF has most influence to the bliss and SA has minimum effect.

6 Conclusion

In this paper, the necessary conditions that were suggested by Cobb and Shenoy are studied for nine different mappings from DST to PT. The results indicate that only the normalized plausibility transformation can meet four conditions among five and the rest of mappings satisfy none of the conditions. Another important point is that the condition of invariance with respect to the marginalization process does not exist for any mappings. In this study, we took a closer look at the projection method in DST, finding

Table 8 mutual ambiguity of the paired variables of Example 9.

	(SA,SB)	(HF,SB)	(FE,SB)	(FS,SB)	(P,SB)
$MAM(X;Y)$	0.0060	0.1757	0.0255	0.1218	0.1371
$AM^{Pl\dot{P}_m}(X;Y)$	0.0044	0.1457	0.0192	0.1210	0.1152

that some probabilistic information is lost in the marginalization process. This problem solved by introducing a Projection Set to retain the probabilistic information. Then, $Bet\acute{P}_m$ and $Pl\acute{P}_m$ which are invariant under the marginalization process was proposed. Similar to the AM that uses the pignistic probability, these modified mappings were utilized in two new ambiguity measures called MAM and $AM^{Pl\acute{P}_m}$. MAM and $AM^{Pl\acute{P}_m}$ against AM are subadditive because $Bet\acute{P}_m$ and $Pl\acute{P}_m$ are independent from the marginalization process. Based on MAM and $AM^{Pl\acute{P}_m}$, the concept of mutual ambiguity were defined in DST. As an application, the mutual ambiguity measures, $MAM(X;Y)$ and $AM^{Pl\acute{P}_m}(\mathrm{X};\mathrm{Y})$ are employed in a social bliss problem to compute the dependency of the variables to the person's happiness. According to many applications of the mutual information in PT, these mutual measures can be used in the future by researchers in various applications.

Appendix A

Proof of proposition 1:

We must prove that $\sum_{j=1}^{|Y|} Bet\acute{P}_{m_{XY}}\big((x_1,y_j)\big) = Bet\acute{P}_{m_X}(x_1)$.We start from the left of term,

$$\sum_{j=1}^{|Y|} Bet\acute{P}_{m_{XY}}\big((x_1,y_j)\big) = \sum_{j=1}^{|Y|} \sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A}} \frac{m^{\Omega_{XY}}(A)}{|A|}$$

$$= \sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=1}} \frac{m^{\Omega_{XY}}(A)}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=|Y|}} \frac{m^{\Omega_{XY}}(A)}{|A|} \right)$$

$$= \sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=1}} \frac{m^{\Omega_{XY}}(A)}{|A|} \right) + \cdots + \sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=|Y|}} \frac{m^{\Omega_{XY}}(A)}{|A|} \right)$$

$$= \sum_{B\subseteq\Omega_X,x_1\in B} \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=1}} \frac{m^{\Omega_{XY}}(A)}{|A|} \quad \text{(A1)}$$

$$+2\sum_{B\subseteq\Omega_X,x_1\in B} \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=2}} \frac{m^{\Omega_{XY}}(A)}{|A|} + \cdots + |Y| \sum_{B\subseteq\Omega_X,x_1\in B} \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=|Y|}} \frac{m^{\Omega_{XY}}(A)}{|A|}$$

$$= \sum_{B\subseteq\Omega_X,x_1\in B} \sum_{A\in PrSt_{\downarrow B}} \frac{m^{\Omega_{XY}}(A).\#(x_1\in A)}{|A|} = Bet\acute{P}_{m_X}(x_1)$$

In line 3, for the first term we have,

$$\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=1}} \frac{m^{\Omega_{XY}}(A)}{|A|} \right) = \sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=1}} \frac{m^{\Omega_{XY}}(A)}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_{|Y|})\in A,\#(x_1\in A)=1}} \frac{m^{\Omega_{XY}}(A)}{|A|} = \sum_{B\subseteq\Omega_X,x_1\in B} \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=1}} \frac{m^{\Omega_{XY}}(A)}{|A|} \quad \text{(A2)}$$

For the second term we have,

$$\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=2}} \frac{m^{\Omega_{XY}}(A)}{|A|} \right) = \sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=2}} \frac{m^{\Omega_{XY}}(A)}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_{|Y|})\in A,\#(x_1\in A)=2}} \frac{m^{\Omega_{XY}}(A)}{|A|} = 2\sum_{B\subseteq\Omega_X,x_1\in B} \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=2}} \frac{m^{\Omega_{XY}}(A)}{|A|} \quad \text{(A3)}$$

This equation is explained with following example:

If $\Omega_{\mathrm{X}} = \{x_1, x_2\}$ and $\Omega_{\mathrm{Y}} = \{y_1, y_2\}$ are the state spaces of X and Y, the joint state space in DST is $2^{\Omega_{\mathrm{XY}}}$ and has $2^4 = 16$ members. For simplicity we use another notation as follows: $\Omega_{XY} = \Omega_X \times \Omega_Y = \{\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{21}, \mathrm{Z}_{22}\}$. We have two term in the right side of above equation as follows (because, $|Y| = 2$):

$$\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=2}}\frac{m^{\Omega_{XY}}(A)}{|A|}+\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_2)\in A,\#(x_1\in A)=2}}\frac{m^{\Omega_{XY}}(A)}{|A|} \tag{A4}$$

Then, we must compute the summation of $m^{\Omega_{XY}}(A)$ in two subsets α and β such that:

$$\alpha=\{A|A\subseteq\Omega_{XY},(x_1,y_1)\in A,\#(x_1\in A)=2\}=\begin{Bmatrix}\{Z_{11},Z_{12}\},\{Z_{11},Z_{12},Z_{21}\},\{Z_{11},Z_{12},Z_{22}\},\\ \{Z_{11},Z_{12},Z_{21},Z_{22}\}\end{Bmatrix} \tag{A5}$$

and,

$$\beta=\{A|A\subseteq\Omega_{XY},(x_1,y_2)\in A,\#(x_1\in A)=2\}=\begin{Bmatrix}\{Z_{11},Z_{12}\},\{Z_{11},Z_{12},Z_{21}\},\{Z_{11},Z_{12},Z_{22}\},\\ \{Z_{11},Z_{12},Z_{21},Z_{22}\}\end{Bmatrix} \tag{A6}$$

It is clear that,

$$\alpha=\beta=\{A|\,A\in PrSt_{\downarrow B}\ and\ \#(x_1\in A)=2\ and\ B\subseteq\Omega_X,x_1\in B\} \tag{A7}$$

So, this two terms is equal to,

$$2\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=2}}\frac{m^{\Omega_{XY}}(A)}{|A|} \tag{A8}$$

Similar to before for the third term we have,

$$\begin{aligned}\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=|Y|}}\frac{m^{\Omega_{XY}}(A)}{|A|}\right)&=\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=|Y|}}\frac{m^{\Omega_{XY}}(A)}{|A|}+\cdots\\&+\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_{|Y|})\in A,\#(x_1\in A)=|Y|}}\frac{m^{\Omega_{XY}}(A)}{|A|}\\&=|Y|\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=|Y|}}\frac{m^{\Omega_{XY}}(A)}{|A|}\end{aligned} \tag{A9}$$

So, the modified pignistic probability is invariant under the marginalization process.

Appendix B

Proof of proposition 2:

We must prove that $\sum_{j=1}^{|Y|}P\acute{l}P_{m_{XY}}((x_1,y_j))=P\acute{l}P_{m_X}(x_1)$.We start from left of term,

$$\begin{aligned}\sum_{j=1}^{|Y|}P\acute{l}P_{m_{XY}}((x_1,y_j))&=\sum_{j=1}^{|Y|}\frac{1}{\Delta}\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A}}m^{\Omega_{XY}}(A)\\&=\frac{1}{\Delta}\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=1}}m^{\Omega_{XY}}(A)+\cdots+\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=|Y|}}m^{\Omega_{XY}}(A)\right)\\&=\frac{1}{\Delta}\left(\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=1}}m^{\Omega_{XY}}(A)\right)+\cdots+\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=|Y|}}m^{\Omega_{XY}}(A)\right)\right)\\&=\frac{1}{\Delta}\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=1}}m^{\Omega_{XY}}(A)+2\frac{1}{\Delta}\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=2}}m^{\Omega_{XY}}(A)+\cdots\\&+|Y|\frac{1}{\Delta}\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=|Y|}}m^{\Omega_{XY}}(A)\\&=\frac{1}{\Delta}\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{A\in PrSt_{\downarrow B}}m^{\Omega_{XY}}(A).\#(x_1\in A)\\&=P\acute{l}P_{m_X}(x_1)\end{aligned} \tag{B1}$$

In line 3, for the first term we have,

$$\begin{aligned}\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=1}}m^{\Omega_{XY}}(A)\right)&=\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=1}}m^{\Omega_{XY}}(A)+\cdots\\&+\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_{|Y|})\in A,\#(x_1\in A)=1}}m^{\Omega_{XY}}(A)\\&=\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\ \#(x_1\in A)=1}}m^{\Omega_{XY}}(A)\end{aligned} \tag{B2}$$

For the second term we have

$$\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=2}} m^{\Omega_{XY}}(A)\right)$$
$$=\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=2}} m^{\Omega_{XY}}(A)+\cdots$$
$$+\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_{|Y|})\in A,\#(x_1\in A)=2}} m^{\Omega_{XY}}(A) \quad (B3)$$
$$=2\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\\#(x_1\in A)=2}} m^{\Omega_{XY}}(A)$$

This equation is explained with the similar example given in the appendix A.

If $\Omega_X=\{x_1,x_2\}$ and $\Omega_Y=\{y_1,y_2\}$ are the state spaces of X and Y, the joint state space in DST is $2^{\Omega_{XY}}$ and has $2^4=16$ members. For simplicity we use another notation as follows: $\Omega_{XY}=\Omega_X\times\Omega_Y=\{Z_{11},Z_{12},Z_{21},Z_{22}\}$. We have two term in right side of above equation as follows (because, $|Y|=2$):

$$\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=2}} m^{\Omega_{XY}}(A)$$
$$+\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_2)\in A,\#(x_1\in A)=2}} m^{\Omega_{XY}}(A) \quad (B4)$$

Then, we must compute the summation of $m^{\Omega_{XY}}(A)$ in two subsets α and β such that:

$$\alpha=\{A|A\subseteq\Omega_{XY},(x_1,y_1)\in A,\#(x_1\in A)=2\}$$
$$=\begin{Bmatrix}\{Z_{11},Z_{12}\},\{Z_{11},Z_{12},Z_{21}\},\{Z_{11},Z_{12},Z_{22}\},\\\{Z_{11},Z_{12},Z_{21},Z_{22}\}\end{Bmatrix} \quad (B5)$$

And,

$$\beta=\{A|A\subseteq\Omega_{XY},(x_1,y_2)\in A,\#(x_1\in A)=2\}$$
$$=\begin{Bmatrix}\{Z_{11},Z_{12}\},\{Z_{11},Z_{12},Z_{21}\},\{Z_{11},Z_{12},Z_{22}\},\\\{Z_{11},Z_{12},Z_{21},Z_{22}\}\end{Bmatrix} \quad (B6)$$

It is clear that,

$$\alpha=\beta=\{A|\,A\in PrSt_{\downarrow B}\ and\ \#(x_1\in A)=2\ and\ B\subseteq\Omega_X,x_1\in B\} \quad (B7)$$

So, this two terms is equal to,

$$2\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\\#(x_1\in A)=2}} m^{\Omega_{XY}}(A) \quad (B8)$$

Similar to before we have,

$$\sum_{j=1}^{|Y|}\left(\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_j)\in A,\#(x_1\in A)=|Y|}} m^{\Omega_{XY}}(A)\right)$$
$$=\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_1)\in A,\#(x_1\in A)=|Y|}} m^{\Omega_{XY}}(A)+\cdots$$
$$+\sum_{\substack{A\subseteq\Omega_{XY}\\(x_1,y_{|Y|})\in A,\#(x_1\in A)=|Y|}} m^{\Omega_{XY}}(A) \quad (B9)$$
$$=|Y|\sum_{B\subseteq\Omega_X,x_1\in B}\ \sum_{\substack{A\in PrSt_{\downarrow B}\\\#(x_1\in A)=|Y|}} m^{\Omega_{XY}}(A)$$

So, the modified normalized plausibility transformation is invariant under the marginalization process.

References

[1] J. Sudano, "Yet Another Paradigm Illustrating Evidence Fusion", *In Proceedings of Fusion 2006 International Conference on Information Fusion*, Firenze, Italy, 2006.

[2] F. Cuzzolin, "Two new Bayesian approximations of belief functions based on convex geometry", *IEEE Trans. on Systems, Man, and Cybernetics part B*, Vol. 37, No. 4, pp. 993-1008, 2007.

[3] P. Smets, "Constructing the pignistic probability function in a context of uncertainty", *M. Henrion, R. D. Shachter, L. N. Kanal, J. F. Lemmer (Eds.), Uncertainty in Artificial Intelligence*, Vol. 5, pp. 29–40, 1990.

[4] P. Smets, "Decision making in a context where uncertainty is represented by belief functions", *R. P. Srivastava, T. J. Mock (Eds.), Belief Functions in Business Decisions*, pp. 316–332, 2002.

[5] P. Smets, "Decision making in the TBM: the necessity of the pignistic transformation", *International Journal of Approximate Reasoning*, Vol. 38, No. 2, pp. 133–147, 2005.

[6] P. Smets and R. Kennes, "The transferable belief model", *Artificial Intelligence*, Vol. 66, No. 2, pp. 191–234, 1994.

[7] F. Smarandache and J. Dezert, *Applications and Advances of DSmT for Information Fusion,* American Research Press, Rehoboth, Vol. 3, 2009.

[8] G. J. Klir, *Uncertainty and Information: Foundations of Generalized Information Theory*, Hoboken, NJ: Wiley Inter science. J., 2006.

[9] G. J. Klir and M. J. Wierman, *Uncertainty-Based Information*, Studies in Fuzziness and Soft Computing, Vol. 15, 1999.

[10] Y. Maeda, H. T. Nguyen and H. Ichihashi, "Maximum entropy algorithms for uncertainty measures", *International Journal of Uncertainty,*

Fuzziness & Knowledge-Based Systems, Vol. 1, No. 1, pp. 69-93, 1993.

[11] D. Harmanec and G. J. Klir, "Measuring total uncertainty in Dempster-Shafer theory", *International Journal of General Systems*, Vol. 22, No. 4, pp. 405-419, 1994.

[12] J. Abellán and S. Moral, "Completing a total uncertainty measure in the Dempster–Shafer theory", *International Journal of General Systems*, Vol. 28, No. 4–5, pp. 299–314, 1999.

[13] A. L. Jousselme, C. Liu, D. Grenier and É. Bossé, "Measuring ambiguity in the evidence theory", *IEEE Transactions on Systems, Man and Cybernetics part A: Systems and Humans*, Vol. 36, No. 5, pp. 890–903, 2006.

[14] G. J. Klir and H. W. Lewis, "Remarks on measuring ambiguity in the evidence theory", *IEEE Transactions on Systems, Man and Cybernetics part A: Systems and Humans*, Vol. 38, No. 4, pp. 995-999, 2008.

[15] B. R. Cobb and P. P. Shenoy, "A comparison of Bayesian and belief function reasoning", *Information Systems Frontiers*, Vol. 5, No. 4, pp. 345–358, 2003.

[16] B. R. Cobb and P. P. Shenoy, "On transforming belief function models to probability models", *Working Paper, school of Business, University of Kansas, Lawrence, KS*, No. 293, 2003.

[17] B. R. Cobb and P. P. Shenoy, "A comparison of methods for transforming belief function models to probability models", *In T. D. Nielsen, N. L. Zhang, eds., Symbolic and Quantitative Approaches to Reasoning with Uncertainty, Lecture Notes in Artificial Intelligence*, Vol. 2711, pp. 255–266, 2003.

[18] B. R. Cobb and P. P. Shenoy, "On The Plausibility Transformation Method for Translating Belief function Models to Probability Models", *Int. J. of Approximation Reasoning*, Vol. 41, No. 3, pp. 314-330, 2006.

[19] A. Dempster, "Upper and lower probabilities induced by multivalued mapping", *Annals of Mathematical Statistics*, Vol. 38, No. 2, pp. 325–339, 1967.

[20] G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, 1976.

[21] A. Dempster, "New methods of reasoning toward posterior distributions based on sample data", *Annals of Mathematical Statistics,* Vol. 37, No. 1966, pp. 355-374, 1987.

[22] J. Sudano, "Pignistic Probability Transforms for Mixes of Low and High Probability Events", *In Proceedings of Fusion 2001 Int. Conf. on Information Fusion, Canada*, August 2001.

[23] F. Voorbraak, "A computationally efficient approximation of Dempster-Shafer theory", *International Journal of Man-Machine Studies*, Vol. 30, No. 5, pp. 525–536, 1989.

[24] C. E. Shannon, "A mathematical theory of communication", *The Bell System Technical Journal*, Vol. 27, No. 3, pp. 379-423, 1948.

[25] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Upper Saddle River, NJ: Prentice-Hall, 1995.

[26] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley, New York. 1990.

[27] A. Shahpari and S. A. Seyedin, "Using Mutual Aggregate Uncertainty Measures in a Threat Assessment Problem Constructed by Dempster-Shafer Network", *IEEE Transactions on Systems, Man and Cybernetics*, Vol. 44, No. 12, pp. 1, 2014.

[28] A. Benavoli, B. Ristic, A. Farina and L. Chisci, "An application of evidential networks to threat assessment", *IEEE Trans. Aerospace and Electronic Systems*, Vol. 45, No. 2, 2009.

[29] B. Ristic and P. Smets, "Target identification using belief functions and implication rules", *IEEE Trans. Aerospace and Electronic Systems*, Vol. 41, No. 3, pp. 1097–1103, 2005.

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