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Two-dimensional time domain fundamental solution to dynamic unsaturated poroelasticity

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Abstract

This paper presents time domain fundamental solutions for the extended Biot's dynamic formulations of two-dimensional (2D) unsaturated poroelasticity. Unsaturated porous media is considered as a porous media in which the voids are saturated with two immiscible fluids, i.e. liquid and gas. At first, the corresponding explicit Laplace transform domain fundamental solution is obtained in terms of skeleton displacements, as well as liquid and gas pressures. Subsequently, the closed-form time domain fundamental solutions are derived by analytical inversion of the Laplace transform domain solutions. Finally, a set of numerical results are presented which verifies the accuracy of the analytically inversed transient fundamental solution and demonstrates some salient features of the elastic waves in unsaturated media.

Keywords: *Unsaturated poroelastodynamics, Wave propagation, Fundamental solution, Boundary element method, Two-dimensional problem.*

1. Introduction

The theory of elasticity for single-fluid saturated porous media was presented in a series of publications by Biot [1,2,3,4] standing on the concepts and principles of continuum mechanics. This theory which ignores the microscopic level and assumes that measurable macroscopic values of classic continuum mechanics are still relevant, was generalized in the context of thermodynamics of open continua to include inelastic behavior and also chemically active unsaturated porous media [5,6,7]. In this extension which is the bases of the governing equations that are used in this paper, unsaturated porous medium is considered as the superposition of several interacting continua in time and space that overlap in a representative elementary volume.

Following to his works, Biot presented the theory of elastic wave propagation in isotropic porous solid saturated by a viscous fluid [8,9]. This theory explains the existence of three body waves propagating in the single fluid saturated porous media. These waves are two compressional waves (P1 & P2-wave), and a shear wave (S-wave). After three decades, Berryman et al. [10] attempted to extend the theory that admits water and air into the voids under the assumption that the wavelength of the excitation is long enough that the capillary pressure changes are negligible. They confirmed the existence of three body waves in the double fluid-saturated porous solid. Later, Wei and Muraleetharan [11] utilized the theory of mixtures with interfaces and expressed acoustic waves in unsaturated porous medium. They showed, there exist three compressional waves instead of two, which the third one will vanish at limiting case of single fluid saturated medium. In a recent study, unsaturated poromechanics was used to derive the wave equations and obtain a full extension of Biot's theory of elastic waves in unsaturated porous solids with and without dissipation at low frequency range [12]. This extension, confirms the existence of three compressional waves and one shear wave propagating in the unsaturated porous solid so that the second and third waves are highly attenuating and vice versa the attenuation of the first compressional wave and the shear wave are negligible.

Fortunately, with the advent of high-speed digital computers, more complex engineering analyses can be performed via computational methods like FEM, BEM, FDM. With the recent growing interest in the boundary element method and its application to the various branches of applied mechanics and appearance of comprehensive

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domain type numerical methods for unsaturated poromechanics, there is a need for determination of transient fundamental solution for dynamic unsaturated poroelasticity for boundary type methods.

History of fundamental solutions for saturated poroelasticity starts with fundamental solutions for the corresponding quasistatic problem were derived by Cleary [13] following from the earlier work of Nowascki [14] in thermoelasticity, while closed-form Laplace domain quasistatic poroelastic fundamental solutions were obtained by Cheng and Liggett [15,16]. However, it seems that the first attempt to obtain fundamental solutions for dynamic poroelasticity was made by Burridge and Vargas [17], who used the saddle point method to obtain displacements at large distances due to a point force in the solid. Later, Norris [18] derived time harmonic Green functions for a point force in the solid and a point force in the fluid. He also obtained explicit asymptotic approximations for far-field displacements, as well as those for high and low frequency responses. Progressively, Kaynia and Banerjee [19] used a solution scheme similar to that of Norris [18] and derived the fundamental solution in the Laplace transform domain as well as transient short-time solution.

Biot had formulated the dynamic poroelasticity in terms of two displacement fields, namely those of the solid skeleton and those of the liquid. For the numerical solution of practical boundary value problems, however, it is more convenient to use the displacement components of the solid and the pressure in the fluid. For this reason, Biot's equations are sometimes cast in terms of these quantities. This reformulation can, however, be achieved only in a transformed domain precisely, and with assumptions in time domain. The advantage of the so called *u-p* formulation is that the resulting coupled equations resemble those of dynamic thermoelasticity for which the Green functions were available [14]. This formulation has been used by Bonnet [20] and Boutin et al. [21] to derive steady-state Green's function of poroelasticity by Kupradze [22] Method. Kaynia [23] adopted a similar approach to derive explicit expressions in Laplace transform domain for Green's functions of dynamic poroelasticity for suddenly applied point force in the solid and a sudden injection of fluid into pores. The errors in Bonnet's paper have been shown by Dominguez [24,25] who set and applied the formulation for the time harmonic saturated poroelastic problems. To fulfill the absence of the fundamental solutions to be applied to the Biot's full dynamic equations, Chen [26,27] presented explicit Laplace transform and approximate transient two and three-dimensional fundamental solution of Biot's full dynamic poroelasticity. Further works have been done on transient fundamental solutions of saturated dynamic poroelasticity based on Zienkiewicz and Shiomi's [28] *u-p* reformulation of the Biot's equations in time domain for medium speed phenomena [29], and also on incorporating incompressibility of solid matrix and liquid compared to compressibility of the skeleton [30,31,32]. Further details on the fundamental solutions of saturated poroelastodynamics can be found in [33].

By appearance of new static and dynamic problems of unsaturated porous media in science and engineering fields like geophysics, geomechanics, geotechnical and environmental engineering, a growing need to develop numerical methods for these problems was found. Two and three dimensional time domain fundamental solutions for quasi-static unsaturated soils were presented by Gatmiri and Jabbari [34,35]. They used the state variables of mean net stress and soil suction in order to represent generalized elastic constitutive relation for soil skeleton. The fundamental solutions were presented in terms of soil skeleton displacement as well as water and air pressures. In a recent publication, Maghoul et al. [36] presented the three-dimensional time domain coupled thermo-hydro-mechanical fundamental solution for the same quasi-static loading condition of unsaturated soils.

So far as the authors know, attempts to find full dynamic fundamental solution of unsaturated porous media were started by Ashayeri et al. [37,38]. They derived the governing boundary integral equation as well as the 2D explicit Laplace transform fundamental solution of full dynamic unsaturated poroelasticity in terms of skeleton displacement and liquid and gas pressures. More recently, analytical 3D transient elastodynamic fundamental solution for unsaturated soils was presented by Ashayeri et al. [39]. The main aim of this paper is to extend the previous works to the time domain and to present the closed-form 2D transient fundamental solution for the full dynamic unsaturated poroelasticity.

2. Governing Equations

The governing differential equations are derived by considering the whole media as superposition of three continuous media in time and space. These continuous media are skeleton, liquid, and gas. The skeleton is composed of solid matrix and empty connected pores. Liquid and gas are filling the pores with the remaining space without the solid matrix. The basic equations of mass balance of constituents, momentum balance of whole mixture and constituents and liquid and gas mass conduction laws are comprehensively presented by Coussy [5,6]. In the following the final equations are summarized for an infinitesimal isothermal transformation. Furthermore, as will be explained later the solid matrix of the skeleton is assumed incompressible compared to the skeleton. It is worth noting that the summation convention is used in the formulations.

Liquid and Gas mass balance

$$(1+\varepsilon)\phi^{lq} D^{lq^{-1}} \dot{p}^{lq} + (1+\varepsilon)\dot{\phi}^{lq} + \phi^{lq}\dot{u}_{i,i} + q^{lq}_{i,i} = \gamma^{lq} \tag{1}$$

$$(1+\varepsilon)\phi^{g} D^{g^{-1}} \dot{p}^{g} - (1+\varepsilon)\dot{\phi}^{lq} + (1-\phi^{lq})\dot{u}_{i,i} + q^{g}_{i,i} = \gamma^{g} \tag{2}$$

where $\phi^{\alpha}_{\alpha=lq,g}$ is the fluid phase "α" volume fraction; $\varepsilon = u_{i,i}$ is the skeleton volume dilatation or volumetric strain; $p^{\alpha}_{\alpha=lq,g}$ is the liquid or gas pressure; $D^{\alpha}_{\alpha=lq,g}$ is the bulk modulus of the liquid or gas; u_i is the displacement vector of the skeleton particle; $q_i^{\alpha}\Big|_{\alpha=lq,g}$ is the liquid or gas flux vector and $\gamma_{\alpha=lq,g}^{\alpha}$ is the rate of liquid or gas injection to the medium.

Momentum balance for whole mixture

$$\sigma_{ij,j} + f_i - \rho\ddot{u}_i - \rho^{lq}\phi^{lq}\ddot{u}_i^{lq} - \rho^{g}\phi^{g}\ddot{u}_i^{g} = 0 \tag{3}$$

where $\rho = \rho^m(1-\phi) + (\rho^{\alpha}\phi^{\alpha})_{\alpha=lq,g}$ is the apparent mass density of the whole medium, $\rho_{\alpha=m,lq,g}^{\alpha}$ is the intrinsic mass density of the solid matrix, liquid or gas, $\phi = \phi^{lq} + \phi^{g}$ is the total porosity, f_i is the body force density per volume unit, $u_i^{\alpha}\Big|_{\alpha=lq,g}$ is the relative displacement vector of the liquid or gas particle with respect to the skeleton particle, and σ_{ij} is the total stress tensor.

Fluid mass conduction laws (generalized Darcy law)

$$q_i^{\alpha} = \phi^{\alpha}\dot{u}_i^{\alpha} = k^{\alpha}(-p_{,i}^{\alpha} - \rho^{\alpha}\ddot{u}_i - \rho^{\alpha}\ddot{u}_i^{\alpha}) \qquad \alpha = lq, g \tag{4}$$

where $k_{\alpha=lq,g}^{\alpha}$ is the isotropic permeability coefficient of the liquid or gas phase.

In addition to eqns. (1) to (4), constitutive relations should be provided for each constituent. The detail of extracting constitutive relations from the first and second thermodynamics laws for unsaturated poroelasticity was presented in [5,6] and the summary of the equations are represented here for an isotropic material:

$$\sigma_{ij} = \lambda\varepsilon\delta_{ij} + 2\mu\varepsilon_{ij} - b^{\alpha}p^{\alpha}\delta_{ij} \qquad \alpha = lq, g \tag{5}$$

$$p^{\alpha} = M^{\alpha\beta}[-(b^{\beta} - \phi^{\beta})\varepsilon - \phi^{\beta}(u_i + u_i^{\beta})_{,i}] \qquad \alpha, \beta = lq, g \tag{6}$$

where λ, μ are drained Lame coefficients, b^{β} is the isotropic Biot's coefficient of fluid phase "β", and $M^{\alpha\beta}$ links the increment of fluid pressure of phase "α" to increment of fluid mass content of phase "β" while the test is undrained with respect to the fluid phase other than "β".

The thermodynamic stability condition of the system implies the following restrictions $(M^{lqg} = M^{glq})$:

$$\sum_{\alpha=lq,g} b^{\alpha} = 1 - \frac{D^{sk}}{D^{m}}, \quad M^{lqlq}M^{gg} - {M^{lqg}}^2 > 0 \tag{7}$$

where D^{sk} and D^{m} are bulk modules of skeleton and solid matrix, respectively.

Usually, it is difficult to perform experimental tests to determine coefficients of eqn (6) for liquid and gas. Meanwhile, assuming solid matrix is incompressible with respect to the porous skeleton (i.e. $D^{sk}/D^{m} \approx 0$) will simplify eqn (6) into one constitutive relation in terms of the capillary pressure and liquid degree of saturation increments. The fundamental solution can be found without this assumption, but due to the difficulty of determination of material properties, the solutions will be very difficult to use or even useless, while with this assumption, tolerable limitation on the generality of the solutions is imposed. Further detail on the extraction of this constitutive relation can be found in [5];

$$dSr^{lq} = \frac{b^{lq} - Sr^{lq}}{(1+\varepsilon)\phi}d\varepsilon + \left(\frac{Sr^{lq}}{D^{lq}} - \frac{N^{lqlq}}{(1+\varepsilon)\phi}\right)dp_c \tag{8}$$

where capillary pressure or suction is defined as $p_c = p^{g} - p^{lq}$, liquid degree of saturation $Sr^{lq} = \phi^{lq}/\phi$, and the square matrix of $[N^{\alpha\beta}]_{2\times2}$ is the inverse of square matrix $[M^{\alpha\beta}]_{2\times2}$.

Equation (7) assures the existence of inverse of $[M^{\alpha\beta}]_{2\times2}$ and the four components of $[N^{\alpha\beta}]_{2\times2}$ are:

$$N^{lqg} = N^{glq} = (1+\varepsilon)\phi^{lq}D^{lq^{-1}} - N^{lqlq} = (1+\varepsilon)\phi^{g}D^{g^{-1}} - N^{gg} \tag{9}$$

Therefore, the final constitutive relations for an isotropic unsaturated linear elastic medium using eqns (5), (8) and definition of liquid degree of saturation are as followings:

Isotropic linear elastic skeleton

$$\sigma_{ij} + p^{g}\delta_{ij} = \lambda\varepsilon\delta_{ij} + 2\mu\varepsilon_{ij} + b^{lq}p_c\delta_{ij} \tag{10}$$

Capillary pressure relation

$$N^{lqg}\dot{p}_c + (b^{lq} - \phi^{lq})\dot{u}_{i,i} = (1+\varepsilon)\dot{\phi}^{lq} \tag{11}$$

One can omit the term $(1+\varepsilon)\dot{\phi}^{lq}$ between eqns (1), (2), and (11):

$$N^{lqg}\dot{p}^{g} + N^{lqlq}\dot{p}^{lq} + b^{lq}\dot{u}_{i,i} + \phi^{lq}\dot{u}_{i,i}^{lq} = \gamma^{lq} \tag{12}$$

$$N^{gg}\dot{p}^{g} + N^{lqg}\dot{p}^{lq} + (1 - b^{lq})\dot{u}_{i,i} + \phi^{g}\dot{u}_{i,i}^{g} = \gamma^{g} \tag{13}$$

Taking the Laplace transform of eqns (3), (4) for both fluids, (12), and (13) with zero initial conditions and performing appropriate substitution, one obtains a new form of the equations in Laplace transform space in terms of spatial derivatives of skeleton displacement $\tilde{u}_i$, liquid and gas pressures $\tilde{p}^{lq}, \tilde{p}^{g}$ that is the precise *u-p* formulation in transformed domain.

$$(\lambda + \mu)\tilde{u}_{j,ji} + \mu\tilde{u}_{i,jj} - \bar{\rho}s^2\tilde{u}_i - \bar{b}^{lq}\tilde{p}_{,i}^{lq} - \bar{b}^{g}\tilde{p}_{,i}^{g} + \tilde{\bar{f}}_i = 0 \tag{14}$$

$$\bar{\eta}^{lq}\tilde{p}_{,ii}^{lq} - N^{lqlq}s\tilde{p}^{lq} - N^{lqg}s\tilde{p}^{g} - \bar{b}^{lq}s\tilde{u}_{i,i} + \tilde{\bar{\gamma}}^{lq} = 0 \tag{15}$$

$$\bar{\eta}^{g}\tilde{p}_{,ii}^{g} - N^{lqg}s\tilde{p}^{lq} - N^{gg}s\tilde{p}^{g} - \bar{b}^{g}s\tilde{u}_{i,i} + \tilde{\bar{\gamma}}^{g} = 0 \tag{16}$$

where s is the Laplace transform parameter and the tilde denotes the Laplace transformation, and $\overline{\eta}^{\alpha} = (1/k^{\alpha} + m^{\alpha} s)^{-1}$, $m^{\alpha} = \rho^{\alpha} / \phi^{\alpha}$, $\overline{b}^{\alpha} = b^{\alpha} - \rho^{\alpha} \overline{\eta}^{\alpha} s$, $\overline{\rho} = \rho - \rho^{lq^2} \overline{\eta}^{lq} s - \rho^{g^2} \overline{\eta}^{g} s$, with $i, j = 1,2$.

Prior to find Laplace transform domain fundamental solutions, it is helpful to present the non-dimensional quantities and governing equations. Thus we define dimensionless coordinates and time by means of:

$$X_i = \frac{x_i}{\rho \overline{K} V_P} \quad \text{and} \quad T = \frac{t}{\rho \overline{K}} \tag{17}$$

where $\overline{K}$ has the physical dimension of permeability and is related to geometrical average of the permeability of the medium with respect to liquid and gas, its value will be discussed more in the numerical demonstration section. And V_P has the dimension of velocity given as:

$$V_P = \sqrt{\frac{H}{\rho}} \quad H = \lambda + 2\mu + b^{\alpha} M^{\alpha\beta} b^{\beta} \Big|_{\alpha,\beta = lq, g} \tag{18}$$

Next, we define a dimensionless skeleton displacement and pore fluids' pressures through:

$$U_i = \frac{u_i}{\rho \overline{K} V_P}, \quad \text{and} \quad P^{\alpha} = \frac{p^{\alpha}}{\rho V_P^2} \quad \alpha = lq, g \tag{19}$$

$$\hat{\lambda} = \frac{\lambda}{H}, \ \hat{\mu} = \frac{\mu}{H}, \ \hat{N}^{\alpha\beta} = N^{\alpha\beta} \times H \quad \alpha = lq, g \tag{20-1}$$

$$\hat{\rho} = 1, \ \hat{\rho}^{\alpha} = \frac{\rho^{\alpha}}{\rho}, \ \hat{m}^{\alpha} = \frac{m^{\alpha}}{\rho}, \ \hat{k}^{\alpha} = \frac{k^{\alpha}}{\overline{K}} \quad \alpha = lq, g \tag{20-2}$$

Therefore, the non-dimensional forms of eqns (14) to (16) are:

$$(\hat{\lambda} + \hat{\mu}) \widetilde{U}_{j,ji} + \hat{\mu} \widetilde{U}_{i,jj} - \hat{\overline{\rho}} s^2 \widetilde{U}_i - \hat{\overline{b}}^{lq} \widetilde{P}^{lq}_{,i} - \hat{\overline{b}}^{g} \widetilde{P}^{g}_{,i} + \widetilde{F}_i = 0 \tag{21}$$

$$\hat{\overline{\eta}}^{lq} \widetilde{P}^{lq}_{,ii} - \hat{N}^{lqlq} s \widetilde{P}^{lq} - \hat{N}^{lqg} s \widetilde{P}^{g} - \hat{\overline{b}}^{lq} s \widetilde{U}_{i,i} + \widetilde{\Gamma}^{lq} = 0 \tag{22}$$

$$\hat{\overline{\eta}}^{g} \widetilde{P}^{g}_{,ii} - \hat{N}^{lqg} s \widetilde{P}^{lq} - \hat{N}^{gg} s \widetilde{P}^{g} - \hat{\overline{b}}^{g} s \widetilde{U}_{i,i} + \widetilde{\Gamma}^{g} = 0 \tag{23}$$

where the parameters are defined as in eqns (14) to (16) but with dimensionless quantities.

Hence, the non-dimensional governing equations and their corresponding fundamental solution take the same form as natural ones.

3. Laplace Transform Domain Fundamental Solution

Fundamental solutions are the response of the medium to point excitation which is a Dirac delta function in space, *δ(x)* and either a Dirac delta function, *δ(t)* or a Heaviside step function in time, *H(t)*. However, for its future application in BEM it is better to consider the solution which results from a Heaviside step function in time. Thus, for a continuous unit line force in the *i*-th direction suddenly applied at the origin, i.e. $f_i(x,t)=\delta(x) \cdot H(t)$, and a unit rate of liquid line injection at the origin, i.e. $\gamma^{lq}(x,t)=\delta(x) \cdot H(t)$, and a unit rate of gas line injection at the origin, i.e. $\gamma^{g}(x,t)=\delta(x) \cdot H(t)$, the Laplace transform of which is $s^{-1}\ \delta(x)$. The two-dimensional Laplace domain fundamental solutions are found by following the Kupradze's Method [22].

It is convenient to write the basic eqns (14) to (16) or the non-dimensional ones eqns (21) to (23) for the two-dimensional case in their matrix form as:

$$B\widetilde{U} + \widetilde{F} = 0 \qquad \widetilde{U}^T = [\widetilde{u}_i \quad \widetilde{p}^{lq} \quad \widetilde{p}^{g}], \qquad \widetilde{F}^T = [\widetilde{f}_i \quad \widetilde{\gamma}^{lq} \quad \widetilde{\gamma}^{g}] \tag{24}$$

where $B_{mn}(\partial x, s)\big|_{4\times4}$ is the differential operator matrix that is defined as follows for $i, j = 1,2$:

$$B_{ij} = (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j} + \delta_{ij} (\mu \nabla^2 - \overline{\rho} s^2), \qquad B_{i3} = -\overline{b}^{lq} \frac{\partial}{\partial x_i}, \quad B_{i4} = -\overline{b}^{g} \frac{\partial}{\partial x_i} \tag{25-1}$$

$$B_{3j} = B_{j3} s, \quad B_{4j} = B_{j4} s, \quad B_{34} = B_{43} = -N^{lqg} s \tag{25-2}$$

$$B_{33} = (\overline{\eta}^{lq} \nabla^2 - N^{lqlq} s), \quad B_{44} = (\overline{\eta}^{g} \nabla^2 - N^{gg} s) \tag{25-3}$$

where ∇^2 is the Laplacian operator.

Therefore, the problem is to find solution matrix $[\widetilde{G}_{ij}]_{4\times4}$, which satisfies:

$$B\widetilde{G} + I s^{-1} \delta(x) = 0 \tag{26}$$

where I is the unit matrix. Following the Kupradze method [22], the fundamental solution is:

$$\widetilde{G} = \overline{B}^T \varphi \tag{27}$$

where $\overline{B}$ is cofactor matrix of B.

Equation (27) enables us to determine the sixteen components of $\widetilde{G}_{mn}\big|_{4\times4}$ by applying the transpose of cofactor matrix to the single scalar function φ. Computing the determinant of differential operator matrix yields

$$\det(B) = \mu(\lambda + 2\mu) \overline{\eta}^{lq} \overline{\eta}^{g} (\nabla^2 - \tfrac{\overline{\rho}}{\mu} s^2) (\nabla^6 + C_4 \nabla^4 + C_2 \nabla^2 + C_0) \tag{28}$$

$$C_4 = -s(f_{40} + f_{41} s) \tag{29-1}$$

$$C_2 = -s^2 (f_{20} + f_{21} s + f_{22} s^2) \tag{29-2}$$

$$C_0 = -s^4 (f_{00} + f_{01} s + f_{02} s^2) \tag{29-3}$$

where the coefficients of eqn (29) are presented in Appendix.

Equation (28) is a polynomial of order four of ∇^2, which one of its roots is the $\lambdabar_s^2 = \bar{\rho} s^2 / \mu$ that $\lambdabar_s$ corresponds to the wave number of the shear wave. The remaining part of eqn (28) is a cubic polynomial in terms of ∇^2, which can have three roots as $\lambdabar_1^2, \lambdabar_2^2, \lambdabar_3^2$ in which $\lambdabar_1, \lambdabar_2, \lambdabar_3$ resemble the wave numbers of compressional waves. After some algebra one can find

$$\varphi = \frac{1}{\mu(\lambda + 2\mu)\bar{\eta}^{lq}\bar{\eta}^{g} s(\lambdabar_3^2 - \lambdabar_s^2)(\lambdabar_2^2 - \lambdabar_1^2)} \left\{ \frac{\varphi_3 - \varphi_2}{(\lambdabar_3^2 - \lambdabar_2^2)} - \frac{\varphi_3 - \varphi_1}{(\lambdabar_3^2 - \lambdabar_1^2)} + \frac{\varphi_s - \varphi_1}{(\lambdabar_s^2 - \lambdabar_1^2)} - \frac{\varphi_s - \varphi_2}{(\lambdabar_s^2 - \lambdabar_2^2)} \right\} \tag{30}$$

where $\varphi_j = \frac{1}{2\pi} K_0(\lambdabar_j r),\ j = 1,2,3,s$ with $r^2 = x_i x_i$ as square of distance of receiver from origin *(0,0),* and $K_0(\lambdabar_j r)$ is the modified Bessel function of second kind of zero order.

Refer to eqn (27) the components of fundamental solution matrix are;

$$\tilde{G}_{44} = \frac{1}{2\pi s \bar{\eta}^{g}} \left\{ \sum_{k=1,2,3} \frac{[(\lambdabar_k^2 - \Pi_{lq})(\lambdabar_k^2 - \Lambda^2) - \bar{B}^{lq}\lambdabar_k^2] K_0(\lambdabar_k r)}{(\lambdabar_{k+1}^2 - \lambdabar_k^2)(\lambdabar_{k+2}^2 - \lambdabar_k^2)} \right\}, \lambdabar_4^2 = \lambdabar_1^2, \lambdabar_5^2 = \lambdabar_2^2 \tag{31-1}$$

$$\tilde{G}_{33} = \frac{1}{2\pi s \bar{\eta}^{lq}} \left\{ \sum_{k=1,2,3} \frac{[(\lambdabar_k^2 - \Pi_{g})(\lambdabar_k^2 - \Lambda^2) - \bar{B}^{g}\lambdabar_k^2] K_0(\lambdabar_k r)}{(\lambdabar_{k+1}^2 - \lambdabar_k^2)(\lambdabar_{k+2}^2 - \lambdabar_k^2)} \right\}, \quad \lambdabar_4^2 = \lambdabar_1^2, \lambdabar_5^2 = \lambdabar_2^2 \tag{31-2}$$

$$\tilde{G}_{43} = \tilde{G}_{34} = \frac{1}{2\pi(\lambda + 2\mu)\bar{\eta}^{lq}\bar{\eta}^{g} s} \left\{ \sum_{k=1,2,3} \frac{[(\lambda + 2\mu)N^{lqg} + \bar{b}^{lq}\bar{b}^{g}] s\lambdabar_k^2 - N^{lqg}\bar{\rho}s^3}{(\lambdabar_{k+1}^2 - \lambdabar_k^2)(\lambdabar_{k+2}^2 - \lambdabar_k^2)} K_0(\lambdabar_k r) \right\}, \qquad \lambdabar_4^2 = \lambdabar_1^2, \lambdabar_5^2 = \lambdabar_2^2 \tag{31-3}$$

$$\tilde{G}_{4j} = \frac{\bar{b}^{g}}{2\pi(\lambda + 2\mu)\bar{\eta}^{g}} \left\{ \sum_{k=1,2,3} \frac{-1}{(\lambdabar_{k+1}^2 - \lambdabar_k^2)(\lambdabar_{k+2}^2 - \lambdabar_k^2)} \left[\lambdabar_k^2 - \frac{(\bar{b}^{g}N^{lqlq} - \bar{b}^{lq}N^{lqg})s}{\bar{b}^{g}\bar{\eta}^{lq}}\right] \frac{\lambdabar_k x_j}{r} K_1(\lambdabar_k r) \right\}, \quad \lambdabar_4^2 = \lambdabar_1^2, \lambdabar_5^2 = \lambdabar_2^2 \tag{31-4}$$

$$\tilde{G}_{3j} = \frac{\bar{b}^{lq}}{2\pi(\lambda + 2\mu)\bar{\eta}^{lq}} \left\{ \sum_{k=1,2,3} \frac{-1}{(\lambdabar_{k+1}^2 - \lambdabar_k^2)(\lambdabar_{k+2}^2 - \lambdabar_k^2)} \left[\lambdabar_k^2 - \frac{(\bar{b}^{lq}N^{gg} - \bar{b}^{g}N^{lqg})s}{\bar{b}^{lq}\bar{\eta}^{g}}\right] \frac{\lambdabar_k x_j}{r} K_1(\lambdabar_k r) \right\}, \quad \lambdabar_4^2 = \lambdabar_1^2, \lambdabar_5^2 = \lambdabar_2^2 \tag{31-5}$$

$$\tilde{G}_{i4} = \frac{1}{s}\tilde{G}_{4j}, \tilde{G}_{i3} = \frac{1}{s}\tilde{G}_{3j} \tag{31-6}$$

$$\tilde{G}_{ij} = \frac{1}{2\pi s} \left\{ \left[\frac{C_{ij}K_0(\lambdabar_s r)}{\mu} - \frac{(\lambdabar_s A_{ij}K_1(\lambdabar_s r) + \lambdabar_s^2 B_{ij}K_0(\lambdabar_s r))}{\bar{\rho}s^2} \right] + \sum_{k=1,2,3} \left[\frac{(\Lambda^2 - \lambdabar_{k+1}^2)(\Lambda^2 - \lambdabar_{k+2}^2) + \Lambda^2(\bar{B}^{lq} + \bar{B}^{g})}{(\lambdabar_{k+1}^2 - \lambdabar_k^2)(\lambdabar_{k+2}^2 - \lambdabar_k^2)} \frac{(\lambdabar_k A_{ij}K_1(\lambdabar_k r) + \lambdabar_k^2 B_{ij}K_0(\lambdabar_k r))}{\bar{\rho}s^2} \right] \right\}, \tag{31-7}$$

$$\lambdabar_4^2 = \lambdabar_1^2, \lambdabar_5^2 = \lambdabar_2^2$$

where $K_1(\lambdabar_i r)$ is the modified Bessel function of second kind of first order and;

$$A_{ij} = \frac{2x_i x_j}{r^3} - \frac{\delta_{ij}}{r}, B_{ij} = \frac{x_i x_j}{r^2}, C_{ij} = \delta_{ij} \qquad i,j = 1,2 \tag{32-1}$$

$$\Pi_{lq} = \frac{N^{lqlq}s}{\bar{\eta}^{lq}}, \Pi_g = \frac{N^{gg}s}{\bar{\eta}^{g}}, \Lambda^2 = \frac{\bar{\rho}s^2}{\lambda + 2\mu}, \bar{B}^{lq} = \frac{\bar{b}^{lq^2}s}{(\lambda + 2\mu)\bar{\eta}^{lq}}, \bar{B}^{g} = \frac{\bar{b}^{g^2}s}{(\lambda + 2\mu)\bar{\eta}^{g}} \tag{32-2}$$

In the above Laplace transform domain solutions, i.e. eqns (31), $\tilde{G}_{ij}$ is the displacement of solid skeleton in *i*-th direction at a point $\xi(x_i,x_j)$ due to the unit Heaviside line force in *j*-th direction at origin. Whereas $\tilde{G}_{3j}$ is the liquid pressure at a point $\xi(x_i,x_j)$ due to the unit Heaviside line force in *j*-th direction at origin. Similarly, $\tilde{G}_{4j}$ is the gas pressure at a point $\xi(x_i,x_j)$ due to the unit Heaviside line force in *j*-th direction at origin. Also $\tilde{G}_{i3}, \tilde{G}_{i4}$ respectively, are the displacements of the solid skeleton in *i*-th direction at a point $\xi(x_i,x_j)$ due to the unit Heaviside rate of liquid and gas line injection at origin. $\tilde{G}_{33}, \tilde{G}_{44}$ are liquid and gas pressure at a point $\xi(x_i,x_j)$ due to the unit Heaviside rate of liquid and gas line injection at origin, respectively. And $\tilde{G}_{34}$ or $\tilde{G}_{43}$ are the liquid or gas pressure at a point $\xi(x_i,x_j)$ due to the unit Heaviside rate of gas or liquid line injection at origin.

4. Transient Fundamental Solution

With the Laplace transform domain fundamental solution being derived, we now proceed to derive its counterpart in the time domain by using analytical inversion. Due to the complexity of the fundamental solution itself and complexity of wave numbers or roots of eqn (28) for unsaturated medium, the analytically inversed time domain solution seems to be extremely difficult. Afterward, we try to reduce the complexity of the analytical inversion problem by means of using appropriate approximation applied on the complex form of wave numbers. These approximations are validated for solutions corresponding to convenient values of Laplace parameter *s*, and more interestingly became more accurate with *s* decreasing that means low frequency range or longer time solutions.

4.1. Discussion on the wave numbers' form

In section 3, it was shown in eqn (28) that the determinant of differential operator matrix is a polynomial of order four in terms of Laplacian operator. Fortunately, the polynomial can be reduced to cubic form easily by separating the wave number corresponding to shear wave, and reduce the problem of finding compressional wave numbers to finding zeros of a cubic polynomial as:

$$\nabla^6 + C_4\nabla^4 + C_2\nabla^2 + C_0 = 0 \tag{33}$$

where the coefficients are introduced in eqns (29-1) to (29-3). Following to the algebraic relation between roots of a cubic polynomial and its coefficients we have:

$$\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2 = -C_4 \tag{33-1}$$

$$\varkappa_1^2\varkappa_2^2 + \varkappa_1^2\varkappa_3^2 + \varkappa_2^2\varkappa_3^2 = C_2 \tag{33-2}$$

$$\varkappa_1^2\varkappa_2^2\varkappa_3^2 = -C_0 \tag{33-3}$$

Now similar to saturated media [27], we assume the wave numbers have the form of:

$$\varkappa_k^2 = c_k^{-2}(s^2 + 2\zeta_k s) = c_k^{-2}[(s+\zeta_k)^2 - \zeta_k^2] \quad k = 1,2,3 \tag{34}$$

where c_k and ζ_k are related to the velocity and attenuation of the k-th compressional wave, respectively. Comparing eqn (33-3) with eqn (29-3) reveals that it is convenient to assume the attenuation of the first compressional wave is negligible compared with the second and third waves i.e. $\zeta_1 \approx 0$. Thus, the wave number of the first compressional wave takes the form of

$$\varkappa_1 = sV_P^{-1} \tag{35}$$

where V_P is the phase wave velocity of fast compressional wave. This is in accordance with the results of calculation of phase wave velocity and attenuation of compressional and shear waves obtained from extended theory of Biot's elastic wave propagation into unsaturated media performed by the authors in a separate research. Ashayeri et al. [12] showed the velocity of the fast compressional wave obtained in absence of dissipation reads:

$$V_P = \sqrt{\frac{\lambda + 2\mu + b^\alpha M^{\alpha\beta} b^\beta}{\rho}} \quad \alpha,\beta = lq, g \tag{36}$$

Introduction of dissipation to the theory at low frequency range shows the velocity of the fast compressional wave changes insignificantly and its attenuation is negligible compared to other compressional waves [13]. It is worth noting that the eqn (36) takes the form of fast compressional wave velocity known in elastodynamics and saturated poroelastodynamics limiting cases with appropriate values of b^α and $M^{\alpha\beta}$.

Back to the eqns (33-1) and (33-2) with keeping in mind ($\zeta_1 \approx 0$) and solving two wave numbers in terms of the third one and using eqns (29-1) to (29-3) gives the following approximation for the velocity and attenuation of the waves i.e. c_k, ζ_k from the current equations:

$$f_{02}c_k^6 + f_{22}c_k^4 + f_{41}c_k^2 - 1 = 0 \qquad k = 1,2,3 \tag{37-1}$$

$$\zeta_k = \frac{f_{40} + f_{21}c_k^2 + f_{01}c_k^4}{2c_k^{-2} + 2f_{22}c_k^2 + 4f_{02}c_k^4} \qquad k = 2,3 \tag{37-2}$$

$$\begin{aligned}&(4f_{22} + 12f_{02}c_k^2)\zeta_k^2 - (2f_{21} + 4f_{01}c_k^2)\zeta_k \\ &+ (f_{20} + f_{00}c_k^2) = 0 \quad k = 2,3\end{aligned} \tag{37-3}$$

Shear wave number is rewritten in terms of Laplace transform parameter s:

$$\varkappa_s = \frac{1}{c_s}\left(s + \frac{\zeta_{s1}s}{s + \beta_1\zeta_{s1}} + \frac{\zeta_{s2}s}{s + \beta_2\zeta_{s2}}\right) \tag{38}$$

$$c_s = \sqrt{\frac{\mu}{\rho}}\left(1 - \frac{1}{2}\left(\frac{\rho^{lq}\phi^{lq}}{\rho} + \frac{\rho^g\phi^g}{\rho}\right)\right)^{-1} \tag{38-1}$$

$$\zeta_{s1} = \frac{{\phi^{lq}}^2}{2\rho k^{lq}}\left(1 - \frac{1}{2}\left(\frac{\rho^{lq}\phi^{lq}}{\rho} + \frac{\rho^g\phi^g}{\rho}\right)\right)^{-1}, \ \beta_1\zeta_{s1} = \frac{\phi^{lq}}{k^{lq}\rho^{lq}} \tag{38-2}$$

$$\zeta_{s2} = \frac{{\phi^g}^2}{2\rho k^g}\left(1 - \frac{1}{2}\left(\frac{\rho^{lq}\phi^{lq}}{\rho} + \frac{\rho^g\phi^g}{\rho}\right)\right)^{-1}, \ \beta_2\zeta_{s2} = \frac{\phi^g}{k^g\rho^g} \tag{38-3}$$

where c_s, ζ_{s1} and ζ_{s2} are related to the shear wave velocity and attenuation, respectively. Theoretical expressions show the shear wave number of unsaturated media in absence of dissipation reads

$$\varkappa_s = \frac{s}{V_S} = \sqrt{\frac{\rho}{\mu}}s \tag{39}$$

Introduction of dissipation in to the theory reveals that the variation of shear wave phase velocity at different frequencies is insignificant and its attenuation is as the same order as the fast compressional wave [13].

4.2. Analytically inversed transient solution

Now we begin to find the analytical inverse of the Laplace transform fundamental solutions using available Laplace transform tables [40]. The most important Laplace transform formulas are listed below:

$$\begin{aligned}&L^{-1}\{K_0(as)\} = \frac{1}{\sqrt{t^2 - a^2}}H(t-a), \\ &L^{-1}\{\frac{1}{s}e^{as}K_1(as)\} = \frac{1}{a}\sqrt{t(t+2a)} \qquad a > 0\end{aligned} \tag{40-1}$$

$$\begin{aligned}&L^{-1}\{K_0(a\sqrt{s^2 - b^2})\} = \frac{1}{\sqrt{t^2 - a^2}} \\ &\cosh(b\sqrt{t^2 - a^2})H(t-a) \qquad a > 0\end{aligned} \tag{40-2}$$

$$\begin{aligned}&L^{-1}\{\frac{1}{\sqrt{s^2 - b^2}}K_1(a\sqrt{s^2 - b^2})\} = \\ &\frac{1}{ab}\sinh(b\sqrt{t^2 - a^2})H(t-a) \qquad a > 0\end{aligned} \tag{40-3}$$

Furthermore, we define the following intermediate function to seek simplicity of expressions:

$$\Gamma_k(x)=\sqrt{x^2-r^2c_k^{-2}} \quad c_1=V_P, c_4=V_S \quad k=1,2,3,4 \tag{41}$$

The coefficients introduced in the following part can be found in Appendix.

Function G_{44}:

Substituting eqn (34) into eqn (31-1) and some algebra one can rewrite eqn (31-1) as:

$$\tilde{G}_{44}=\sum_{k=2,3}\frac{a_{1k3}s^3+a_{1k2}s^2+a_{1k1}s+a_{1k0}}{b_{1k0}s(b_{1k1}s+b_{1k2})(b_{1k3}s+b_{1k4})}K_0(\lambda_k r)+ \frac{a_{112}s^2+a_{111}s+a_{110}}{(b_{111}s+b_{112})(b_{113}s+b_{114})}K_0(\lambda_1 r) \tag{42}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (42) as:

$$\tilde{G}_{44}=\sum_{k=2,3}[e_{1k00}+\frac{e_{1k0}}{s}+\frac{e_{1k1}}{s+c_{1k1}}+\frac{e_{1k2}}{s+c_{1k2}}]K_0(\lambda_k r)+ [e_{110}+\frac{e_{111}}{s+c_{111}}+\frac{e_{112}}{s+c_{112}}]K_0(\lambda_1 r) \tag{43}$$

Thus the G_{44} is calculated using convolution integral as:

$$G_{44}=\{\frac{e_{110}}{\Gamma_1(t)}+\int_{\frac{r}{V_P}}^{t}\frac{(e_{111}e^{-c_{111}(t-\tau)}+e_{112}e^{-c_{112}(t-\tau)})}{\Gamma_1(\tau)}d\tau\}H(t-\frac{r}{V_P})+ \sum_{k=2,3}\{e_{1k00}e^{-\zeta_k t}\frac{\cosh(\zeta_k\Gamma_k(t))}{\Gamma_k(t)}+\int_{\frac{r}{c_k}}^{t}(e_{1k0}+e_{1k1}e^{-c_{1k1}(t-\tau)}+e_{1k2}e^{-c_{1k2}(t-\tau)})e^{-\zeta_k\tau}\frac{\cosh(\zeta_k\Gamma_k(\tau))}{\Gamma_k(\tau)}d\tau\}H(t-\frac{r}{c_k}) \tag{44}$$

Function G_{33}:

Substituting eqn (34) into eqn (31-2) and some algebra one can rewrite eqn (31-2) as:

$$\tilde{G}_{33}=\sum_{k=2,3}\frac{a_{2k3}s^3+a_{2k2}s^2+a_{2k1}s+a_{2k0}}{b_{2k0}s(b_{2k1}s+b_{2k2})(b_{2k3}s+b_{2k4})}K_0(\lambda_k r)+ \frac{a_{212}s^2+a_{211}s+a_{210}}{(b_{211}s+b_{212})(b_{213}s+b_{214})}K_0(\lambda_1 r) \tag{45}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (45) as:

$$\tilde{G}_{33}=\sum_{k=2,3}[e_{2k00}+\frac{e_{2k0}}{s}+\frac{e_{2k1}}{s+c_{2k1}}+\frac{e_{2k2}}{s+c_{2k2}}]K_0(\lambda_k r)+ [e_{210}+\frac{e_{211}}{s+c_{211}}+\frac{e_{212}}{s+c_{212}}]K_0(\lambda_1 r) \tag{46}$$

Thus the G_{33} is calculated using convolution integral as:

$$G_{33}=\{\frac{e_{210}}{\Gamma_1(t)}+\int_{\frac{r}{V_P}}^{t}\frac{(e_{211}e^{-c_{211}(t-\tau)}+e_{212}e^{-c_{212}(t-\tau)})}{\Gamma_1(\tau)}d\tau\}H(t-\frac{r}{V_P})+ \sum_{k=2,3}\{e_{2k00}e^{-\zeta_k t}\frac{\cosh(\zeta_k\Gamma_k(t))}{\Gamma_k(t)}+\int_{\frac{r}{c_k}}^{t}(e_{2k0}+e_{2k1}e^{-c_{2k1}(t-\tau)}+e_{2k2}e^{-c_{2k2}(t-\tau)})e^{-\zeta_k\tau}\frac{\cosh(\zeta_k\Gamma_k(\tau))}{\Gamma_k(\tau)}d\tau\}H(t-\frac{r}{c_k}) \tag{47}$$

Function G_{34}:

Substituting eqn (34) into eqn (31-3) and some algebra one can rewrite eqn (31-3) as:

$$\tilde{G}_{34}=\sum_{k=2,3}\frac{a_{3k3}s^3+a_{3k2}s^2+a_{3k1}s+a_{3k0}}{b_{3k0}s(b_{3k1}s+b_{3k2})(b_{3k3}s+b_{3k4})}K_0(\lambda_k r)+ \frac{a_{312}s^2+a_{311}s+a_{310}}{(b_{311}s+b_{312})(b_{313}s+b_{314})}K_0(\lambda_1 r) \tag{48}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (48) as:

$$\tilde{G}_{34}=\sum_{k=2,3}[e_{3k00}+\frac{e_{3k0}}{s}+\frac{e_{3k1}}{s+c_{3k1}}+\frac{e_{3k2}}{s+c_{3k2}}]K_0(\lambda_k r)+ [e_{310}+\frac{e_{311}}{s+c_{311}}+\frac{e_{312}}{s+c_{312}}]K_0(\lambda_1 r) \tag{49}$$

Thus the G_{34} is calculated using convolution integral as:

$$G_{34}=\{\frac{e_{310}}{\Gamma_1(t)}+\int_{\frac{r}{V_P}}^{t}\frac{(e_{311}e^{-c_{311}(t-\tau)}+e_{312}e^{-c_{312}(t-\tau)})}{\Gamma_1(\tau)}d\tau\}H(t-\frac{r}{V_P})+$$
$$\sum_{k=2,3}\{e_{3k00}e^{-\zeta_k t}\frac{\cosh(\zeta_k\Gamma_k(t))}{\Gamma_k(t)}+\int_{\frac{r}{c_k}}^{t}(e_{3k0}+e_{3k1}e^{-c_{3k1}(t-\tau)}+e_{3k2}e^{-c_{3k2}(t-\tau)})e^{-\zeta_k\tau}\frac{\cosh(\zeta_k\Gamma_k(\tau))}{\Gamma_k(\tau)}d\tau\}H(t-\frac{r}{c_k}) \tag{50}$$

Function G_{4j}:

Substituting eqn (34) into eqn (31-4) and some algebra one can rewrite eqn (31-4) as:

$$\tilde{G}_{4j}=\sum_{k=2,3}\frac{a_{4k3}s^3+a_{4k2}s^2+a_{4k1}s+a_{4k0}}{b_{4k0}(b_{4k1}s+b_{4k2})(b_{4k3}s+b_{4k4})}\frac{1}{\lambda_k}K_1(\lambda_k r)+$$
$$\frac{a_{413}s^3+a_{412}s^2+a_{411}s}{b_{410}(b_{411}s+b_{412})(b_{413}s+b_{414})}\frac{1}{s}K_1(\lambda_1 r) \tag{51}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (51) as:

$$\tilde{G}_{4j}=\sum_{k=2,3}[e_{4k00}s+e_{4k0}+\frac{e_{4k1}}{s+c_{4k1}}+\frac{e_{4k2}}{s+c_{4k2}}]\frac{1}{\lambda_k}K_1(\lambda_k r)+$$
$$[e_{4100}s+e_{410}+\frac{e_{411}}{s+c_{411}}+\frac{e_{412}}{s+c_{412}}]\frac{1}{s}K_1(\lambda_1 r) \tag{52}$$

Thus the G_{4j} is calculated using convolution integral as:

$$G_{4j}=\{e_{4100}\frac{V_P}{r}\frac{t}{\Gamma_1(t)}+e_{410}\frac{V_P}{r}\Gamma_1(t)+\frac{V_P}{r}\int_{\frac{r}{V_P}}^{t}(e_{411}e^{-c_{411}(t-\tau)}+e_{412}e^{-c_{412}(t-\tau)})\Gamma_1(\tau)d\tau\}H(t-\frac{r}{V_P})+$$
$$\sum_{k=2,3}\left[\begin{array}{l}\{e_{4k00}\frac{c_k^2}{r}[e^{-\zeta_k t}\frac{t}{\Gamma_k(t)}\cosh(\zeta_k\Gamma_k(t))-e^{-\zeta_k t}\sinh(\zeta_k\Gamma_k(t))]+e_{4k0}\frac{c_k^2}{r\zeta_k}e^{-\zeta_k t}\sinh(\zeta_k\Gamma_k(t))\\+\frac{c_k^2}{r\zeta_k}\int_{\frac{r}{c_k}}^{t}(e_{4k1}e^{-c_{4k1}(t-\tau)}+e_{4k2}e^{-c_{4k2}(t-\tau)})e^{-\zeta_k\tau}\sinh(\zeta_k\Gamma_k(\tau))d\tau\}H(t-\frac{r}{c_k})\end{array}\right] \tag{54}$$

Function G_{3j}:

Substituting eqn (34) into eqn (31-5) and some algebra one can rewrite eqn (31-5) as:

$$\tilde{G}_{3j}=\sum_{k=2,3}\frac{a_{5k3}s^3+a_{5k2}s^2+a_{5k1}s+a_{5k0}}{b_{5k0}(b_{5k1}s+b_{5k2})(b_{5k3}s+b_{5k4})}\frac{1}{\lambda_k}K_1(\lambda_k r)+$$
$$\frac{a_{513}s^3+a_{512}s^2+a_{511}s}{b_{510}(b_{511}s+b_{512})(b_{513}s+b_{514})}\frac{1}{s}K_1(\lambda_1 r) \tag{54}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (54) as:

$$\tilde{G}_{4j}=\sum_{k=2,3}[e_{5k00}s+e_{5k0}+\frac{e_{5k1}}{s+c_{5k1}}+\frac{e_{5k2}}{s+c_{5k2}}]\frac{1}{\lambda_k}K_1(\lambda_k r)+$$
$$[e_{5100}s+e_{510}+\frac{e_{511}}{s+c_{511}}+\frac{e_{512}}{s+c_{512}}]\frac{1}{s}K_1(\lambda_1 r) \tag{55}$$

Thus the G_{3j} is calculated using convolution integral as:

$$G_{3j}=\{e_{5100}\frac{V_P}{r}\frac{t}{\Gamma_1(t)}+e_{510}\frac{V_P}{r}\Gamma_1(t)+\frac{V_P}{r}\int_{\frac{r}{V_P}}^{t}(e_{511}e^{-c_{511}(t-\tau)}+e_{512}e^{-c_{512}(t-\tau)})\Gamma_1(\tau)d\tau\}H(t-\frac{r}{V_P})+$$
$$\sum_{k=2,3}\left[\begin{array}{l}\{e_{5k00}\frac{c_k^2}{r}[e^{-\zeta_k t}\frac{t}{\Gamma_k(t)}\cosh(\zeta_k\Gamma_k(t))-e^{-\zeta_k t}\sinh(\zeta_k\Gamma_k(t))]+e_{5k0}\frac{c_k^2}{r\zeta_k}e^{-\zeta_k t}\sinh(\zeta_k\Gamma_k(t))\\+\frac{c_k^2}{r\zeta_k}\int_{\frac{r}{c_k}}^{t}(e_{5k1}e^{-c_{5k1}(t-\tau)}+e_{5k2}e^{-c_{5k2}(t-\tau)})e^{-\zeta_k\tau}\sinh(\zeta_k\Gamma_k(\tau))d\tau\}H(t-\frac{r}{c_k})\end{array}\right] \tag{56}$$

Function G_{i4}:

Substituting eqn (34) into eqn (31-6) and some algebra one can rewrite eqn (31-6) as:

$$\tilde{G}_{i4} = \sum_{k=2,3} \frac{a_{6k3}s^3 + a_{6k2}s^2 + a_{6k1}s + a_{6k0}}{b_{6k0}s(b_{6k1}s + b_{6k2})(b_{6k3}s + b_{6k4})} \frac{1}{\lambda_k} K_1(\lambda_k r) + \frac{a_{612}s^2 + a_{612}s + a_{610}}{b_{610}(b_{611}s + b_{612})(b_{613}s + b_{614})} \frac{1}{s} K_1(\lambda_1 r) \tag{57}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (57) as:

$$\tilde{G}_{i4} = \sum_{k=2,3} [e_{6k00} + \frac{e_{6k0}}{s} + \frac{e_{6k1}}{s + c_{6k1}} + \frac{e_{6k2}}{s + c_{6k2}}] \frac{1}{\lambda_k} K_1(\lambda_k r) + [e_{610} + \frac{e_{611}}{s + c_{611}} + \frac{e_{612}}{s + c_{612}}] \frac{1}{s} K_1(\lambda_1 r) \tag{58}$$

Thus the G_{i4} is calculated using convolution integral as:

$$G_{i4} = \{e_{610} \frac{V_P}{r} \Gamma_1(t) + \frac{V_P}{r} \int_{\frac{r}{V_P}}^{t} (e_{611} e^{-c_{611}(t-\tau)} + e_{612} e^{-c_{612}(t-\tau)}) \Gamma_1(\tau) d\tau\} H(t - \frac{r}{V_P}) + \sum_{k=2,3} \frac{c_k^2}{r\zeta_k} \{e_{6k00} e^{-\zeta_k t} \sinh(\zeta_k \Gamma_k(t)) + \int_{\frac{r}{c_k}}^{t} (e_{6k0} + e_{6k1} e^{-c_{6k1}(t-\tau)} + e_{6k2} e^{-c_{6k2}(t-\tau)}) e^{-\zeta_k \tau} \sinh(\zeta_k \Gamma_k(\tau)) d\tau\} H(t - \frac{r}{c_k}) \tag{59}$$

Function G_{i3}:

Substituting eqn (34) into eqn (31-6) and some algebra one can rewrite eqn (31-6) as:

$$\tilde{G}_{i3} = \sum_{k=2,3} \frac{a_{7k3}s^3 + a_{7k2}s^2 + a_{7k1}s + a_{7k0}}{b_{7k0}s(b_{7k1}s + b_{7k2})(b_{7k3}s + b_{7k4})} \frac{1}{\lambda_k} K_1(\lambda_k r) + \frac{a_{712}s^2 + a_{712}s + a_{710}}{b_{710}(b_{711}s + b_{712})(b_{713}s + b_{714})} \frac{1}{s} K_1(\lambda_1 r) \tag{60}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (60) as:

$$\tilde{G}_{i3} = \sum_{k=2,3} [e_{7k00} + \frac{e_{7k0}}{s} + \frac{e_{7k1}}{s + c_{7k1}} + \frac{e_{7k2}}{s + c_{7k2}}] \frac{1}{\lambda_k} K_1(\lambda_k r) + [e_{710} + \frac{e_{711}}{s + c_{711}} + \frac{e_{712}}{s + c_{712}}] \frac{1}{s} K_1(\lambda_1 r) \tag{61}$$

Thus the G_{i3} is calculated using convolution integral as:

$$G_{i3} = \{e_{710} \frac{V_P}{r} \Gamma_1(t) + \frac{V_P}{r} \int_{\frac{r}{V_P}}^{t} (e_{711} e^{-c_{711}(t-\tau)} + e_{712} e^{-c_{712}(t-\tau)}) \Gamma_1(\tau) d\tau\} H(t - \frac{r}{V_P}) + \sum_{k=2,3} \frac{c_k^2}{r\zeta_k} \{e_{7k00} e^{-\zeta_k t} \sinh(\zeta_k \Gamma_k(t)) + \int_{\frac{r}{c_k}}^{t} (e_{7k0} + e_{7k1} e^{-c_{7k1}(t-\tau)} + e_{7k2} e^{-c_{7k2}(t-\tau)}) e^{-\zeta_k \tau} \sinh(\zeta_k \Gamma_k(\tau)) d\tau\} H(t - \frac{r}{c_k}) \tag{62}$$

Function G_{ij}:

The term G_{ij} is the most complex one, since it is affected by all four body waves existing in the medium. The complex form of wave numbers of compressional body waves were simplified in previous part, using the complete form of shear wave number will generate a complex convolution integral in time domain. Therefore, we use the simplified form of eqn (39) instead of eqn (38) for shear wave number to reduce the complexity of the convolution integral. Later in numerical demonstration, the accuracy of this simplification will be verified.

Hence, Substituting eqns (34) and (39) into eqn (31-7) and some algebra one can rewrite eqn (31-7) as:

$$\tilde{G}_{ij} = \frac{C_{ij}}{2\pi\mu} (\frac{1}{s}) K_0(\frac{r}{V_S} s) - \frac{a_{840} + a_{841}s + a_{842}s^2}{(b_{840} + b_{841}s + s^2)s} [\frac{A_{ij}}{2\pi V_S} \frac{1}{s} K_1(\frac{r}{V_S} s) + \frac{B_{ij}}{2\pi V_S^2} K_0(\frac{r}{V_S} s)] + \sum_{k=2,3} (-1)^{k+1} \{\frac{a_{8k2}s^2 + a_{8k1}s + a_{8k0}}{s(b_{8k0}s + b_{8k1})(b_{8k2}s + b_{8k3})} + \frac{a_{9k4}s^4 + a_{9k3}s^3 + a_{9k2}s^2 + a_{9k1}s + a_{9k0}}{s(b_{8k0}s + b_{8k1})(b_{8k2}s + b_{8k3})(b_{840} + b_{841}s + s^2)}\} [\frac{A_{ij}}{2\pi} \frac{1}{\lambda_k} K_1(\lambda_k r) + \frac{B_{ij}}{2\pi} K_0(\lambda_k r)] + \{\frac{a_{811}s + a_{810}}{(b_{811}s + b_{812})(b_{813}s + b_{814})} + \frac{a_{914}s^4 + a_{913}s^3 + a_{912}s^2 + a_{911}s + a_{910}}{s(b_{811}s + b_{812})(b_{813}s + b_{814})(b_{840} + b_{841}s + s^2)}\} [\frac{A_{ij}}{2\pi} \frac{1}{\lambda_1} K_1(\lambda_1 r) + \frac{B_{ij}}{2\pi} K_0(\lambda_1 r)] \tag{63}$$

One can use partial fraction decomposition and determine e_{ijkl} to rewrite eqn (63) as:

$$\widetilde{G}_{ij}=\frac{C_{ij}}{2\pi\mu}(\frac{1}{s})K_0(\frac{r}{V_S}s)-(\frac{e_{840}}{s}+\frac{e_{841}s+e_{842}}{(s+c_{841})^2-c_{840}^2})[\frac{A_{ij}}{2\pi V_S}\frac{1}{s}K_1(\frac{r}{V_S}s)+\frac{B_{ij}}{2\pi V_S^2}K_0(\frac{r}{V_S}s)]+$$
$$\sum_{k=2,3}(-1)^{k+1}\{\frac{e_{8k1}}{s}+\frac{e_{8k2}}{s+c_{8k0}}+\frac{e_{8k3}}{s+c_{8k1}}+\frac{e_{9k1}}{s}+\frac{e_{9k2}}{s+c_{8k0}}+\frac{e_{9k3}}{s+c_{8k1}}+\frac{e_{9k4}s+e_{9k5}}{((s+c_{841})^2-c_{840}^2)}\}[\frac{A_{ij}}{2\pi}\frac{1}{\lambda_k}K_1(\lambda_k r)+\frac{B_{ij}}{2\pi}K_0(\lambda_k r)] \quad (64)$$
$$+\{\frac{e_{810}}{s+c_{810}}+\frac{e_{811}}{s+c_{811}}+\frac{e_{911}}{s}+\frac{e_{912}}{s+c_{810}}+\frac{e_{913}}{s+c_{811}}+\frac{e_{914}s+e_{915}}{((s+c_{841})^2-c_{840}^2)}\}[\frac{A_{ij}}{2\pi}\frac{V_P}{s}K_1(\frac{r}{V_P}s)+\frac{B_{ij}}{2\pi}K_0(\lambda_1 r)]$$

Thus the G_{ij} is calculated using convolution integral as:

$$G_{ij}=\{\frac{C_{ij}}{2\pi\mu}\int_{\frac{r}{V_S}}^{t}\frac{1}{\Gamma_4(\tau)}d\tau-\frac{A_{ij}}{2\pi V_S}[e_{840}\frac{V_S}{r}\int_{\frac{r}{V_S}}^{t}\Gamma_4(\tau)d\tau+$$
$$\frac{V_S}{r}\int_{\frac{r}{V_S}}^{t}(e_{841}\cosh(c_{840}(t-\tau))+\frac{e_{842}-c_{841}e_{841}}{c_{840}}\sinh(c_{840}(t-\tau)))e^{-c_{841}(t-\tau)}\Gamma_4(\tau)d\tau]$$
$$-\frac{B_{ij}}{2\pi V_S^2}[e_{840}\int_{\frac{r}{V_S}}^{t}\frac{1}{\Gamma_4(\tau)}d\tau+\int_{\frac{r}{V_S}}^{t}(e_{841}\cosh(c_{840}(t-\tau))+\frac{e_{842}-c_{841}e_{841}}{c_{840}}\sinh(c_{840}(t-\tau)))\frac{e^{-c_{841}(t-\tau)}}{\Gamma_4(\tau)}d\tau]\}H(t-\frac{r}{V_S})+$$
$$\sum_{k=2,3}\left[\begin{array}{l}(-1)^{k+1}\{\frac{A_{ij}}{2\pi}\frac{c_k^2}{r\zeta_k}[\int_{\frac{r}{c_k}}^{t}((e_{8k1}+e_{9k1})+(e_{8k2}+e_{9k2})e^{-c_{8k0}(t-\tau)}+(e_{8k3}+e_{9k3})e^{-c_{8k1}(t-\tau)})e^{-\zeta_k\tau}\sinh(\zeta_k\Gamma_k(\tau))d\tau+\\ \int_{\frac{r}{c_k}}^{t}(e_{9k4}\cosh(c_{840}(t-\tau))+\frac{e_{9k5}-c_{841}e_{9k4}}{c_{840}}\sinh(c_{840}(t-\tau))]e^{-c_{841}(t-\tau)}e^{-\zeta_k\tau}\sinh(\zeta_k\Gamma_k(\tau)d\tau]+\\ \frac{B_{ij}}{2\pi}[\int_{\frac{r}{c_k}}^{t}((e_{8k1}+e_{9k1})+(e_{8k2}+e_{9k2})e^{-c_{8k0}(t-\tau)}+(e_{8k3}+e_{9k3})e^{-c_{8k1}(t-\tau)})e^{-\zeta_k\tau}\frac{\cosh(\zeta_k\Gamma_k(\tau))}{\Gamma_k(\tau)}d\tau+\\ \int_{\frac{r}{c_k}}^{t}(e_{9k4}\cosh(c_{840}(t-\tau))+\frac{e_{9k5}-c_{841}e_{9k4}}{c_{840}}\sinh(c_{840}(t-\tau)))e^{-c_{841}(t-\tau)}e^{-\zeta_k\tau}\frac{\cosh(\zeta_k\Gamma_k(\tau))}{\Gamma_k(\tau)}d\tau]\}H(t-\frac{r}{c_k})\end{array}\right]+ \quad (65)$$
$$\{\frac{A_{ij}}{2\pi}\frac{V_P^2}{r}[\int_{\frac{r}{V_P}}^{t}(e_{911}+(e_{810}+e_{912})e^{-c_{810}(t-\tau)}+(e_{811}+e_{913})e^{-c_{811}(t-\tau)})\Gamma_1(\tau)d\tau+$$
$$\int_{\frac{r}{V_P}}^{t}(e_{914}\cosh(c_{840}(t-\tau))+\frac{e_{915}-c_{841}e_{914}}{c_{840}}\sinh(c_{840}(t-\tau)))e^{-c_{841}(t-\tau)}\Gamma_1(\tau)d\tau]$$
$$+\frac{B_{ij}}{2\pi}[\int_{\frac{r}{V_P}}^{t}\frac{e_{911}+(e_{810}+e_{912})e^{-c_{810}(t-\tau)}+(e_{811}+e_{913})e^{-c_{811}(t-\tau)}}{\Gamma_1(\tau)}d\tau+$$
$$\int_{\frac{r}{V_P}}^{t}(e_{914}\cosh(c_{840}(t-\tau))+\frac{e_{915}-c_{841}e_{914}}{c_{840}}\sinh(c_{840}(t-\tau)))\frac{e^{-c_{841}(t-\tau)}}{\Gamma_1(\tau)}d\tau]\}H(t-\frac{r}{V_P})$$

5. Numerical Demonstration

Since the analytical expressions of the fundamental solution are extremely complicated, it is tried to investigate the accuracy and features of the solutions by means of a numerical example. In this example the analytically inversed time domain fundamental solutions are compared with their numerically inversed Laplace transform solution counterparts, graphically. Furthermore, without losing generality and to seek simplicity, the solutions are verified for liquid degree of saturation of $Sr^{lq}=0.5$. The example is obtained from experimental works by Murphy [41] that the fast compressional and shear wave velocities and attenuations were measured in various liquid degrees of saturation in Massilon sandstone (the liquid is water and gas is air). This example was used by [10], [11] and [12] to verify their theories. Some of the parameters that are used in this work are constant at all liquid degrees of saturation, which are listed in Table (1) with their values and units. Other parameters are functions of the water degree of saturation or the capillary pressure. Hence, a constitutive relation between water degree of saturation and capillary pressure was assumed as in Fig. (1), this assumption was made so that to be in accordance with the nature and porosity of the Massilon sandstone. In this way, the water degree of saturation corresponding to air-entry suction and residual suction values were considered equal to 0.95 and 0.05 respectively.

Table 1 List of material constants Murphy (1982)

Material Properties	Symbol	Value	Unit
Solid Matrix Bulk Modulus	D^m	3.50×10^{10}	Pa
Water Bulk Modulus	D^{lq}	2.25×10^{9}	Pa
Air Bulk Modulus	D^g	1.45×10^{5}	Pa
Density of Solid Matrix	ρ^m	2650	kg/m^3
Density of Liquid	ρ^{lq}	997	kg/m^3
Density of Gas	ρ^g	1.10	kg/m^3
Porosity	ϕ	0.23	-
Intrinsic Permeability	K	2.50×10^{-12}	m^2
Absolute Viscosity of Liquid	ν^{lq}	1.0×10^{-3}	Pas
Absolute Viscosity of Gas	ν^g	1.8×10^{-5}	Pas

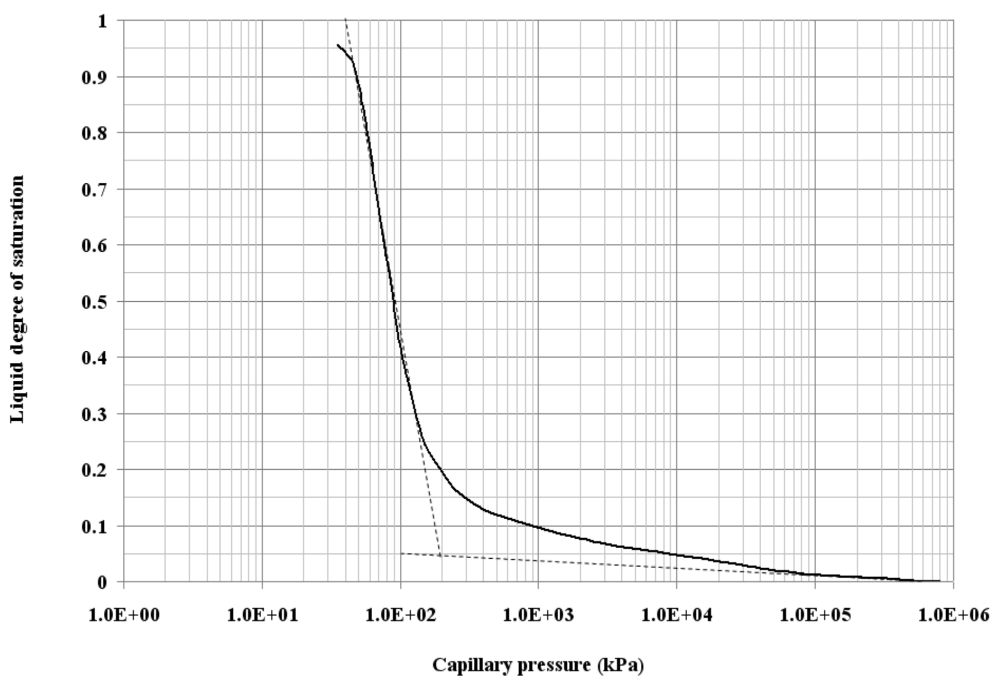

Fig. 1 Constitutive relation between capillary pressure and liquid degree of saturation

Matching this constitutive relation with eqn (8) with $b^{lq} = Sr^{lq}$ and using eqn (9) will give the values of some other physical parameters. Permeability of the medium with respect to water and air are related to the water degree of saturation and capillary pressure through the model proposed by Brooks and Corey [42] with $\theta = 1$ as follows:

$$k^{lq} = \begin{cases} (\frac{K}{\nu^{lq}}) & p_c \le 40kPa \\ (\frac{K}{\nu^{lq}})(\frac{40}{p_c})^{2+3\theta} & p_c > 40kPa \end{cases},$$
$$k^{g} = \begin{cases} (\frac{K}{\nu^{g}})(1 - Sr^{lq})^{2+3\theta} & p_c \le 40kPa \\ (\frac{K}{\nu^{g}})(1 - (\frac{40}{p_c})^{\theta})^2 (1 - (\frac{40}{p_c})^{2+\theta}) & p_c > 40kPa \end{cases} \quad (66)$$

Table (2) summarizes the values of these parameters for water degrees of saturation equal to 0.5.

Table 2 Material parameters and dimensionless quantities used for calculating analytically inversed solutions

Material parameters			Dimensionless quantities	
Symbol	Value	Unit	Symbol	Value
Sr^{lq}	0.5		$\hat{V}_P$	1.0
ϕ^{lq}	0.115	-	$\hat{c}_2$	0.3107
ϕ^{g}	0.115	-	$\hat{c}_3$	6.77×10^{-3}
b^{lq}	0.5	-	$\hat{\zeta}_2$	0.8822
b^{g}	0.5	-	$\hat{\zeta}_3$	0.7944
N^{lqlq}	1.84×10^{-6}	Pa^{-1}	$\hat{V}_s$	0.6997
k^{lq}	4.68×10^{-11}	m^2/Pas	T_P	2.236
k^{g}	3.80×10^{-8}	m^2/Pas	T_2	7.197
λ	6.0×10^{7}	Pa	T_3	330.291
μ	1.45×10^{9}	Pa	T_S	3.196

It was noticed in order to have the results in a meaningful manner, the equations and quantities should be non-dimensional. Thus, by means of eqns (17) to (20) the dimensionless quantities are calculated and used in numerical demonstration. Since the variation of the permeability of medium with respect to liquid and gas are very large (four orders of magnitude), a model for variation of $\overline{K}$ similar to Brooks and Corey's model is used:

$$\overline{K} = \begin{cases} \left(\dfrac{K}{v^{lq}}\right) & p_c \le 40kPa \\ \sqrt{k^{lq}k^{g}}\left(1-\dfrac{40}{p_c}\right)^{\frac{2+3\theta}{2}} & p_c > 40kPa \end{cases} \tag{67}$$

Numerical inversion of the Laplace transform fundamental solution was calculated by DINLAP subroutine of Fortran Power Station 4.0, which is based on applying the epsilon algorithm to the complex Fourier series obtained as a discrete approximation to the inversion integral [43]. As previously discussed, the fast compressional wave number was assumed as eqn (35). Therefore, non-dimensional eqn (33) is reduced one order by eqn (35) as:

$$\nabla^4 + (\hat{C}_4 + \hat{\lambda}_1^2)\nabla^2 + (\hat{C}_2 + \hat{C}_4\hat{\lambda}_1^2 + \hat{\lambda}_1^4) = 0 \tag{68}$$

And the second and the third compressional wave with shear wave numbers are:

$$\hat{\lambda}_{2,3}^2 = \frac{1}{2}[-(\hat{C}_4 + \hat{\lambda}_1^2) \mp \Delta], \quad \Delta = \sqrt{(\hat{C}_4 + \hat{\lambda}_1^2)^2 - 4(\hat{C}_2 + \hat{C}_4\hat{\lambda}_1^2 + \hat{\lambda}_1^4)} \tag{69}$$

$$\hat{\lambda}_s = \sqrt{\frac{\hat{\bar{\rho}}}{\hat{\mu}}}s \tag{70}$$

The analytically inversed time domain fundamental solution was found by calculating integrals using accurate numerical program with the non-dimensional wave numbers of eqns (34), (35), and (39). Table (2) summarizes the dimensionless quantities used for calculating integrals.

Figures (2) to (10) show some components of two-dimensional fundamental solution. Note that other components differ in the direction with the following ones, and for the verification of the analytical solution it is enough to show the accuracy of the analytical solution of the following components. The applied force point or the liquid or gas source point is located at origin *(0,0)* and the receiver is chosen at the non-dimensional coordinate *(1,2)*. Referring to dimensionless wave velocities in table (2), the non-dimensional time required for the four waves to reach receiver are calculated (Table 2). The non-dimensional time required for the fast compressional wave to reach the receiver point, T_P = 2.236 is clearly found on the all the components of the fundamental solution. It is mostly observed as an impulse in the magnitude of the component. Shortly after the fast compressional wave, the shear wave arrives to the receiver, T_S = 3.196, but it is not observable in pressure components of the fundamental solution due to disability of the shear wave to propagate in liquid and gas.

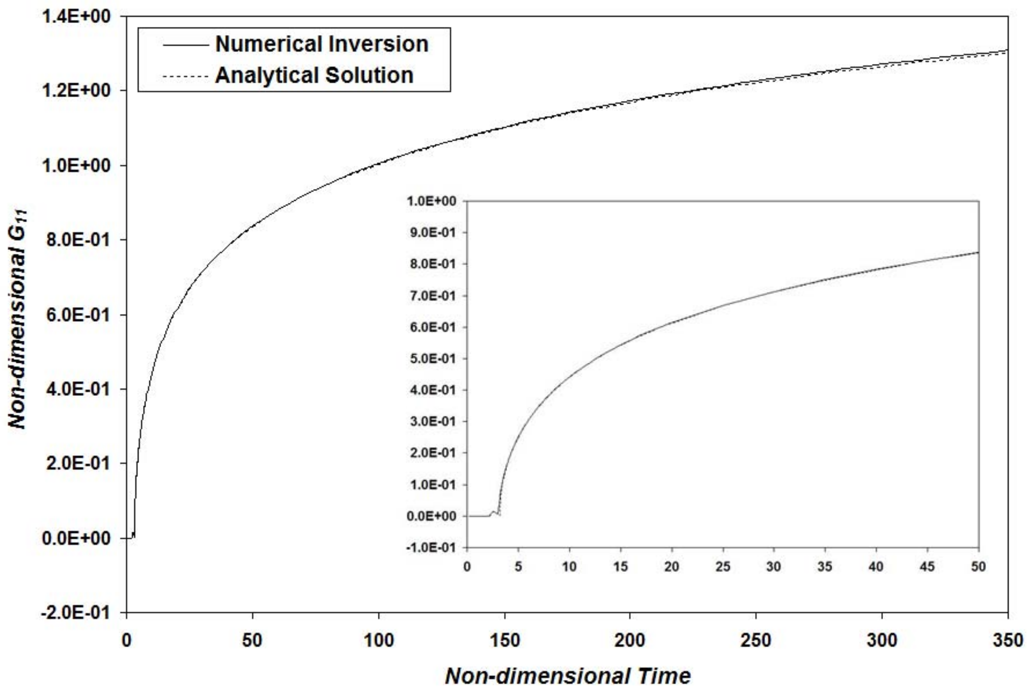

Fig. 2 Non-dimensional G_{11} at (1,2) for Sr^{Jq}=0.5

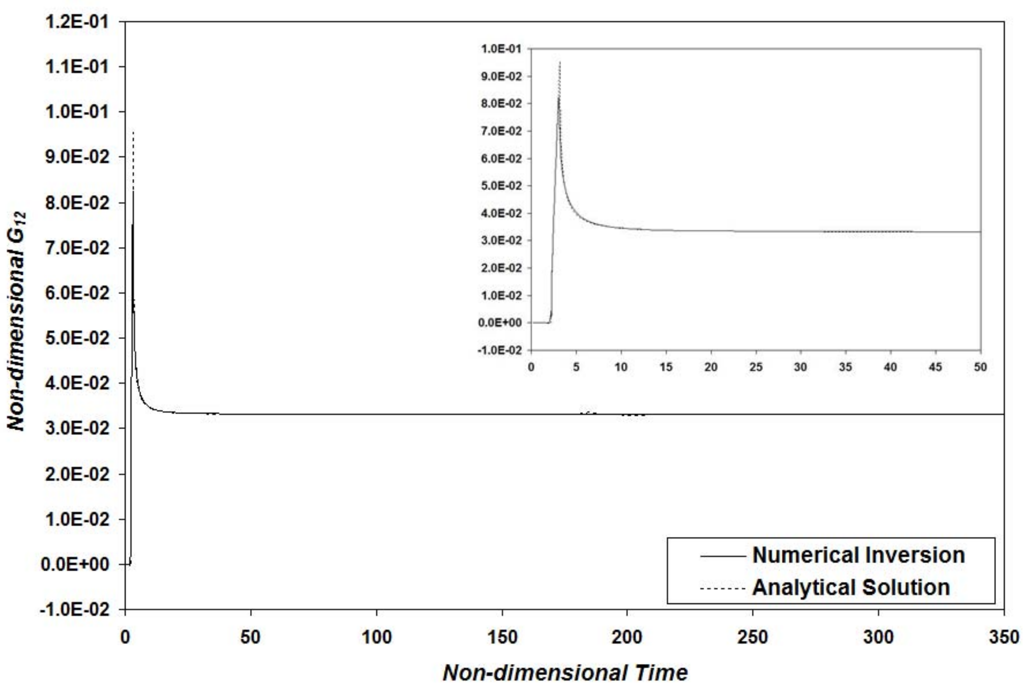

Fig. 3 Non-dimensional G_{12} at (1,2) for Sr^{Jq}=0.5

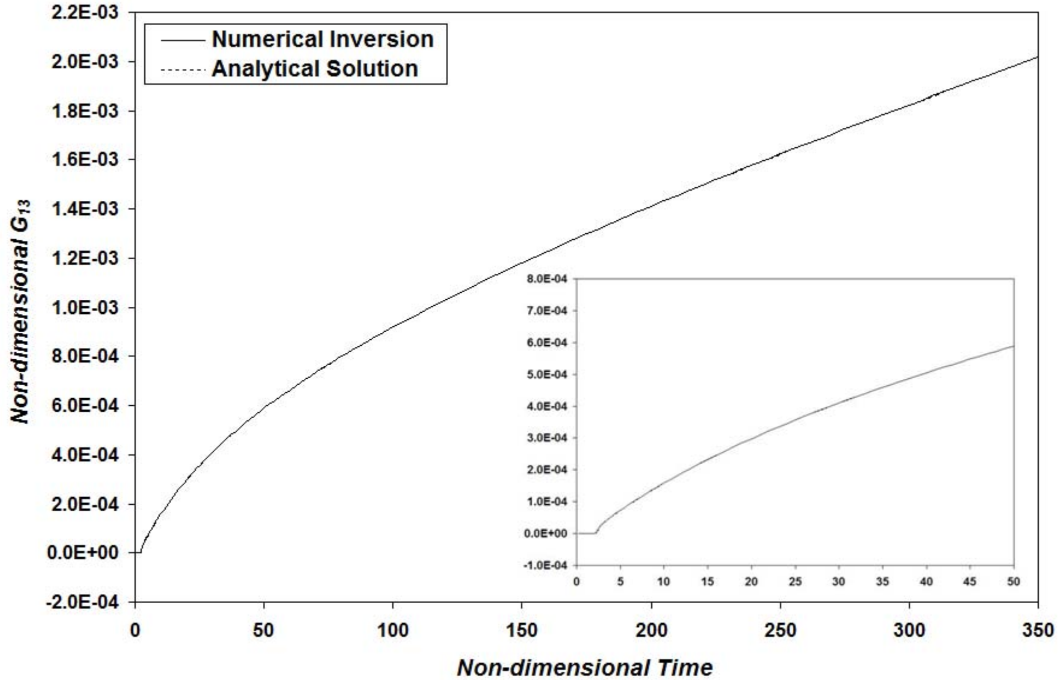

Fig. 4 Non-dimensional G_{13} at (1,2) for Sr^{Jq}=0.5

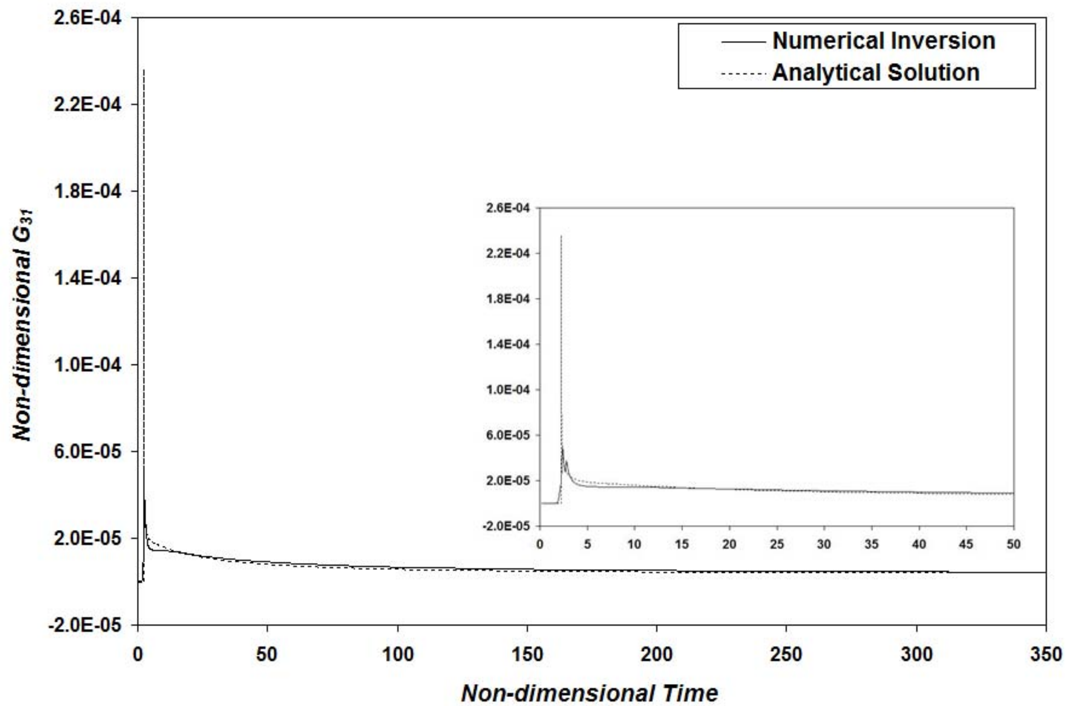

Fig. 5 Non-dimensional G_{31} at (1,2) for Sr^{Jq}=0.5

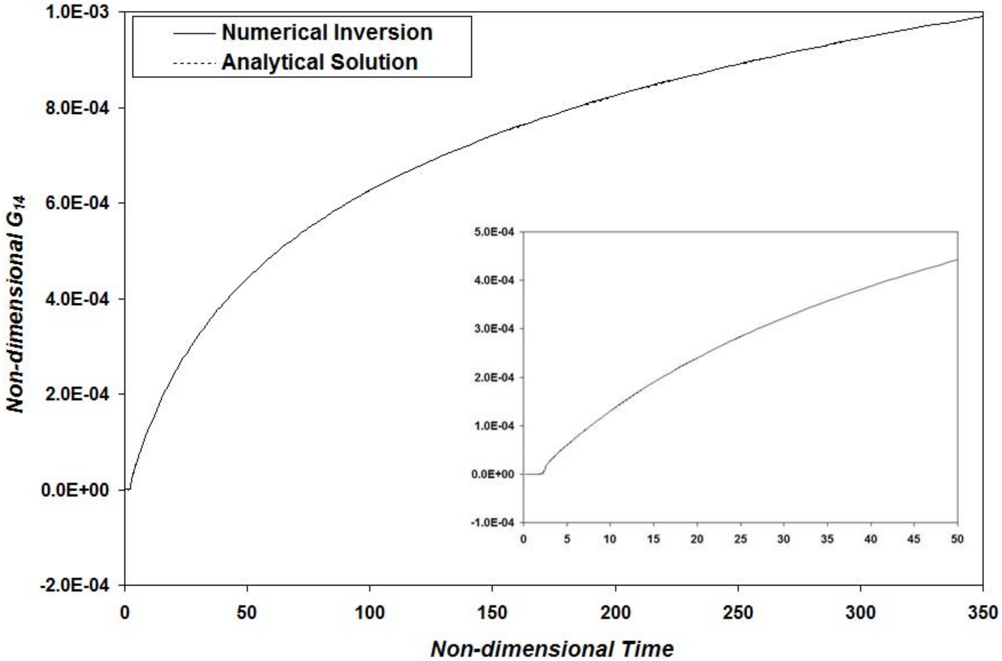

Fig. 6 Non-dimensional G_{14} at (1,2) for Sr^{lq}=0.5

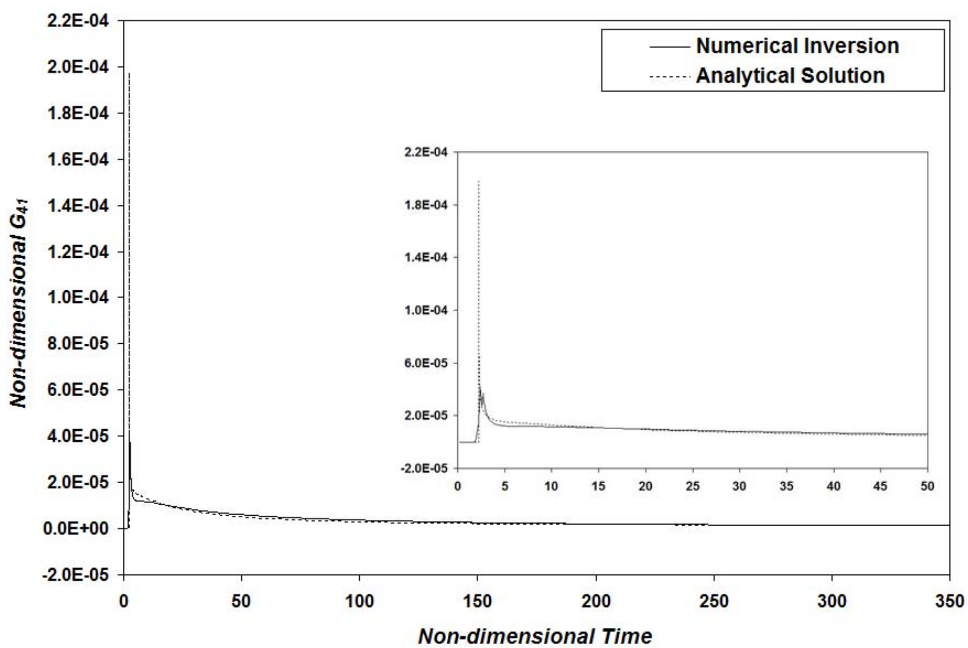

Fig. 7 Non-dimensional G_{41} at (1,2) for Sr^{lq}=0.5

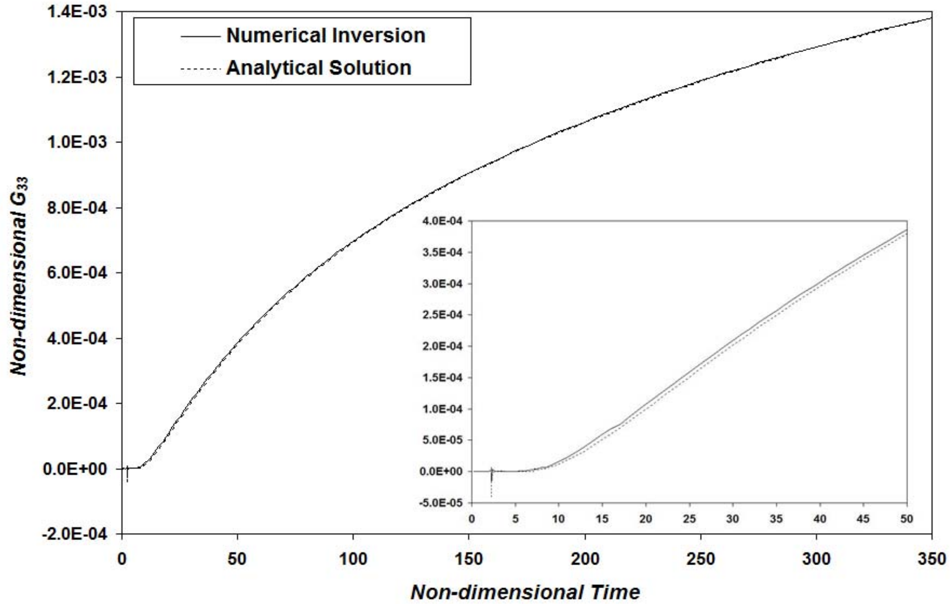

Fig. 8 Non-dimensional G_{33} at (1,2) for Sr^{Jq}=0.5

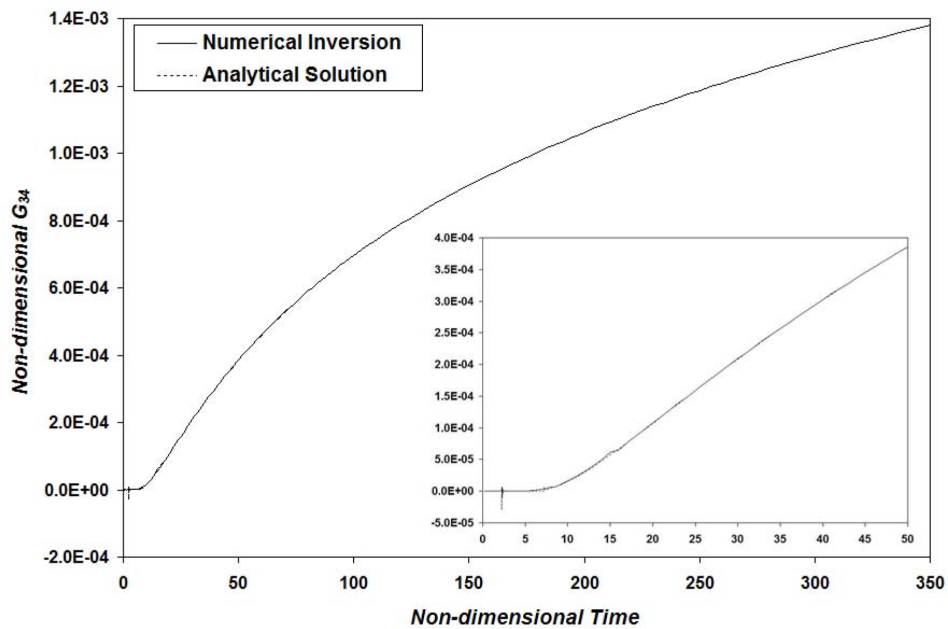

Fig. 9 Non-dimensional G_{34} at (1,2) for Sr^{Jq}=0.5

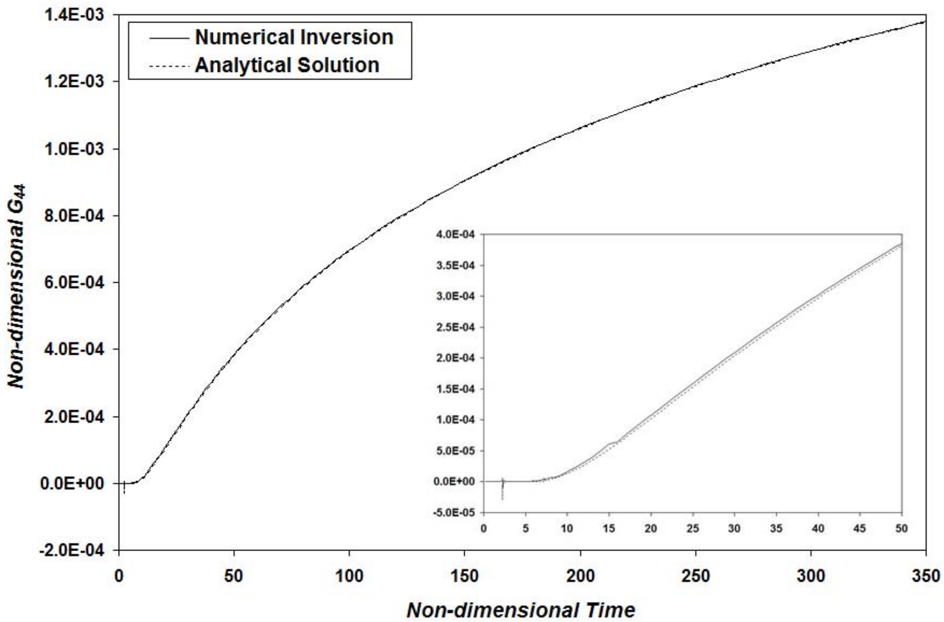

Fig. 10 Non-dimensional G_{44} at (1,2) for Sr^{lq}=0.5

Figures (2) to (10) all support the theory that second and third compressional waves are of diffusive wave kind and are strongly attenuated with high dissipative factors (table 2). With respect to the arrival times of the second and third compressional waves, T_2 = 7.197 and T_3 = 330.291 respectively, the amplitude of these waves were attenuated so strongly that gave no tangible contribution to the magnitude of the components. It seems that the second compressional wave front is detected by G_{33}, G_{44} and G_{34}, but no significant contribution was given to G_{11} and G_{12}. The highly dissipative nature of the third compressional wave results no contribution of this wave to the components of the fundamental solution of receiver point at *(1,2)*. The wave front of the third compressional wave might be observed at receivers very close to the origin, but it is strongly attenuated from the origin and it contribution can be neglected at short distances from the origin.

To compare the results of this study with the 3D-fundamental solution of the same medium given in [39], G_{11} at the receiver point *(1,2,3)* is represented in Fig. (11). Same features of 2D-fundamental solution are observed in the 3D-one, like observation of the fast compressional and shear wave fronts. However, the general shape is different because of the dimension of the problem. The 2D-fundamental solution is obtained for the line force in space, while the 3D-fundamental solution is obtained for the point force in space. In the 2D-fundamental solution the unit Heaviside force in the origin is extended over the third axis, hence the effects of infinite number of point forces along the third axis are integrated. This results in gradually increasing magnitude of displacement to the limiting value at longer times and higher magnitude of the total displacement of 2D-fundamental solution.

[Fig. 11]

Generally speaking, excellent agreement of analytically inversed solution and numerical Laplace inversion is seen. The agreement between analytically inversed solution and numerical inversion for displacement component of fundamental solution (Figs. 2, 3) justifies the simplifying assumption of neglecting shear wave attenuation. Figures (8) to (10) present the fact that the pressure solutions are not affected by the shear wave and pressure change due to line source of liquid or gas at origin reach to a steady state at longer times in the receiver point.

6. Conclusion

This paper presented the closed-form time domain fundamental solution for the 2D dynamic unsaturated poroelasticity. Based on the key fact of the Biot's theory, it was assumed that the fast compressional wave is true wave and its attenuation is practically negligible. In order to reduce the complexity of the convolution integrals in the displacement component of the transient fundamental solution, the attenuation of the shear wave was assumed negligible too.

A set of numerical results was presented, which verifies the accuracy of the assumptions made for the analytically inversed transient fundamental solution and demonstrates some salient features of elastic waves propagating in unsaturated porous media.

The presented transient fundamental solution enables the future development of an efficient time domain BEM in order to solve various wave propagation problems in linear unsaturated poroelastic media as well as hybrid FEM-BEM to solve non-linear unsaturated poroelastic media where the far field is formulated by the boundary elements and the near field by the non-linear finite elements.

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Appendix

Coefficients of eqn (29):

$$f_{41}=\frac{N^{gg}\rho^{g}}{\phi^{g}}+\frac{N^{lqlq}\rho^{lq}}{\phi^{lq}}+\frac{\rho}{\lambda+2\mu}+\frac{\rho^{lq}b^{lq^2}}{(\lambda+2\mu)\phi^{lq}}-\frac{2\rho^{lq}b^{lq}}{\lambda+2\mu}+\frac{\rho^{g}b^{g^2}}{(\lambda+2\mu)\phi^{g}}-\frac{2\rho^{g}b^{g}}{\lambda+2\mu}$$

$$f_{40}=\frac{N^{gg}}{k^{g}}+\frac{N^{lqlq}}{k^{lq}}+\frac{b^{lq^2}}{(\lambda+2\mu)k^{lq}}+\frac{b^{g^2}}{(\lambda+2\mu)k^{g}}$$

$$f_{22}=\frac{\rho^{lq}\rho^{g}}{\phi^{lq}\phi^{g}}(N^{lqg^2}-N^{lqlq}N^{gg})-\frac{N^{lqlq}}{\lambda+2\mu}(\frac{\rho\rho^{lq}}{\phi^{lq}}-\rho^{lq^2})-\frac{N^{gg}}{\lambda+2\mu}(\frac{\rho\rho^{g}}{\phi^{g}}-\rho^{g^2})$$
$$-\frac{\rho^{lq}\rho^{g}}{\lambda+2\mu}(\frac{b^{lq^2}N^{gg}+b^{g^2}N^{lqlq}-2N^{lqg}b^{lq}b^{g}}{\phi^{lq}\phi^{g}}-\frac{2b^{lq}N^{gg}}{\phi^{g}}+\frac{2N^{lqg}b^{lq}}{\phi^{lq}}+\frac{2N^{lqg}b^{g}}{\phi^{g}}-2N^{lqg}-\frac{2b^{g}N^{lqlq}}{\phi^{lq}})$$

$$f_{21}=(\frac{\rho^{g}}{k^{lq}\phi^{g}}+\frac{\rho^{lq}}{k^{g}\phi^{lq}})(N^{lqg^2}-N^{lqlq}N^{gg}-\frac{b^{lq^2}N^{gg}}{\lambda+2\mu}+\frac{2N^{lqg}b^{lq}b^{g}}{\lambda+2\mu}-\frac{b^{g^2}N^{lqlq}}{\lambda+2\mu})-\frac{\rho}{\lambda+2\mu}(\frac{N^{lqlq}}{k^{lq}}+\frac{N^{gg}}{k^{g}})$$
$$+\frac{1}{\lambda+2\mu}(\frac{2\rho^{lq}b^{lq}N^{gg}}{k^{g}}-\frac{2N^{lqg}b^{lq}\rho^{g}}{k^{lq}}-\frac{2N^{lqg}b^{g}\rho^{lq}}{k^{g}}+\frac{2\rho^{g}b^{g}N^{lqlq}}{k^{lq}})$$

$$f_{20}=\frac{1}{k^{lq}k^{g}}[N^{lqg^2}-N^{lqlq}N^{gg}-\frac{1}{\lambda+2\mu}(b^{lq^2}N^{gg}-2N^{lqg}b^{lq}b^{g}-b^{g^2}N^{lqlq})]$$

$$f_{02}=\frac{N^{lqlq}N^{gg}-N^{lqg^2}}{\lambda+2\mu}(\frac{\rho\rho^{lq}\rho^{g}}{\phi^{lq}\phi^{g}}-\frac{\rho^{lq^2}\rho^{g}}{\phi^{g}}-\frac{\rho^{g^2}\rho^{lq}}{\phi^{lq}})$$

$$f_{01}=\frac{N^{lqlq}N^{gg}-N^{lqg^2}}{\lambda+2\mu}(\frac{\rho\rho^{g}}{k^{lq}\phi^{g}}+\frac{\rho\rho^{lq}}{k^{g}\phi^{lq}}-\frac{\rho^{lq^2}}{k^{g}}-\frac{\rho^{g^2}}{k^{lq}})$$

$$f_{00}=\frac{N^{lqlq}N^{gg}-N^{lqg^2}}{\lambda+2\mu}\frac{\rho}{k^{lq}k^{g}}$$

Coefficients of G_{ij} matrix

$$a_{112}=\frac{1}{2\pi}[\frac{\rho^{g}}{\phi^{g}V_P^2}(\frac{1}{V_P^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{lq}\rho^{lq}}{\lambda+2\mu})+\frac{\rho^{g^2}}{\lambda+2\mu}\frac{1}{V_P^2}-\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^{g}}{\phi^{g}}(\frac{N^{lqlq}}{V_P^2}-\frac{\rho N^{lqlq}}{\lambda+2\mu}+\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{V_P^2})-\frac{\rho^{lq}\rho^{g}N^{lqlq}}{\lambda+2\mu}(\frac{\rho^{lq}}{\phi^{g}}+\frac{\rho^{g}}{\phi^{lq}})]$$

$$a_{111}=\frac{1}{2\pi}[\frac{1}{V_P^2k^{g}}(\frac{1}{V_P^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{lq}\rho^{lq}}{\lambda+2\mu})-\frac{N^{lqlq}}{\lambda+2\mu}(\frac{\rho^{lq^2}}{k^{g}}+\frac{\rho^{g^2}}{k^{lq}})-(\frac{1}{k^{lq}}\frac{\rho^{g}}{\phi^{g}}+\frac{1}{k^{g}}\frac{\rho^{lq}}{\phi^{lq}})(\frac{N^{lqlq}}{V_P^2}-\frac{\rho N^{lqlq}}{\lambda+2\mu}+\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{V_P^2})]$$

$$a_{110}=-\frac{1}{2\pi}\frac{1}{k^{lq}k^{g}}(\frac{N^{lqlq}}{V_P^2}-\frac{\rho N^{lqlq}}{\lambda+2\mu}+\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{V_P^2})$$

$$a_{1k3}=\frac{\rho^{g}}{\phi^{g}c_k^2}(\frac{1}{c_k^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{lq}\rho^{lq}}{\lambda+2\mu})+\frac{\rho^{g^2}}{\lambda+2\mu}\frac{1}{c_k^2}-\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^{g}}{\phi^{g}}(\frac{N^{lqlq}}{c_k^2}-\frac{\rho N^{lqlq}}{\lambda+2\mu}+\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{c_k^2})-\frac{\rho^{lq}\rho^{g}N^{lqlq}}{\lambda+2\mu}(\frac{\rho^{lq}}{\phi^{g}}+\frac{\rho^{g}}{\phi^{lq}})$$

$$a_{1k2}=\frac{1}{c_k^2k^{g}}(\frac{1}{c_k^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{lq}\rho^{lq}}{\lambda+2\mu})+\frac{2\zeta_k}{c_k^2}\frac{\rho^{g}}{\phi^{g}}(\frac{2}{c_k^2}+\frac{2b^{lq}\rho^{lq}}{\lambda+2\mu})-\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^{g}}{\phi^{g}}\frac{2\zeta_k}{c_k^2}(N^{lqlq}+\frac{b^{lq^2}}{\lambda+2\mu})$$
$$-\frac{1}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}(\frac{\rho\rho^{g}}{\phi^{g}}-\rho^{g^2})-(\frac{1}{k^{lq}}\frac{\rho^{g}}{\phi^{g}}+\frac{1}{k^{g}}\frac{\rho^{lq}}{\phi^{lq}})(\frac{N^{lqlq}}{c_k^2}-\frac{\rho N^{lqlq}}{\lambda+2\mu}+\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{c_k^2})-\frac{N^{lqlq}}{\lambda+2\mu}(\frac{\rho^{lq^2}}{k^{g}}+\frac{\rho^{g^2}}{k^{lq}})$$

$$a_{1k1}=\frac{2\zeta_k}{c_k^2}\frac{1}{k^{g}}(\frac{2}{c_k^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{lq}\rho^{lq}}{\lambda+2\mu})-\frac{1}{k^{lq}k^{g}}(\frac{N^{lqlq}}{c_k^2}-\frac{\rho N^{lqlq}}{\lambda+2\mu}+\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{c_k^2})-\frac{2\zeta_k}{c_k^2}(\frac{1}{k^{lq}}\frac{\rho^{g}}{\phi^{g}}+\frac{1}{k^{g}}\frac{\rho^{lq}}{\phi^{lq}})(N^{lqlq}+\frac{b^{lq^2}}{\lambda+2\mu})+\frac{4\zeta_k^2}{c_k^4}\frac{\rho^{g}}{\phi^{g}}$$

$$a_{1k0}=\frac{2\zeta_k}{c_k^2}(\frac{2\zeta_k}{c_k^2}\frac{1}{k^{g}}-\frac{N^{lqlq}}{k^{lq}k^{g}}-\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{k^{lq}k^{g}})$$

$$a_{212}=\frac{1}{2\pi}[\frac{\rho^{lq}}{\phi^{lq}V_P^2}(\frac{1}{V_P^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{g}\rho^{g}}{\lambda+2\mu})+\frac{\rho^{lq^2}}{\lambda+2\mu}\frac{1}{V_P^2}-\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^{g}}{\phi^{g}}(\frac{N^{gg}}{V_P^2}-\frac{\rho N^{gg}}{\lambda+2\mu}+\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{V_P^2})-\frac{\rho^{lq}\rho^{g}N^{gg}}{\lambda+2\mu}(\frac{\rho^{lq}}{\phi^{g}}+\frac{\rho^{g}}{\phi^{lq}})]$$

$$a_{211}=\frac{1}{2\pi}(\frac{1}{V_P^2k^{lq}}(\frac{1}{V_P^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{g}\rho^{g}}{\lambda+2\mu})-\frac{N^{gg}}{\lambda+2\mu}(\frac{\rho^{lq^2}}{k^{g}}+\frac{\rho^{g^2}}{k^{lq}})-(\frac{1}{k^{lq}}\frac{\rho^{g}}{\phi^{g}}+\frac{1}{k^{g}}\frac{\rho^{lq}}{\phi^{lq}})(\frac{N^{gg}}{V_P^2}-\frac{\rho N^{gg}}{\lambda+2\mu}+\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{V_P^2})$$

$$a_{210}=-\frac{1}{2\pi}\frac{1}{k^{lq}k^{g}}(\frac{N^{gg}}{V_P^2}-\frac{\rho N^{gg}}{\lambda+2\mu}+\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{V_P^2})$$

$$a_{2k3}=[\frac{1}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}}(\frac{1}{c_k^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{g}\rho^{g}}{\lambda+2\mu})+\frac{\rho^{lq^2}}{\lambda+2\mu}\frac{1}{c_k^2}-\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^{g}}{\phi^{g}}(\frac{N^{gg}}{c_k^2}-\frac{\rho N^{gg}}{\lambda+2\mu}+\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{c_k^2})-\frac{\rho^{lq}\rho^{g}N^{gg}}{\lambda+2\mu}(\frac{\rho^{lq}}{\phi^{g}}+\frac{\rho^{g}}{\phi^{lq}})]$$

$$a_{2k2}=\frac{1}{c_k^2k^{lq}}(\frac{1}{c_k^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{g}\rho^{g}}{\lambda+2\mu})+\frac{2\zeta_k}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}}(\frac{2}{c_k^2}+\frac{2b^{g}\rho^{g}}{\lambda+2\mu})-\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^{g}}{\phi^{g}}\frac{2\zeta_k}{c_k^2}(N^{gg}+\frac{b^{g^2}}{\lambda+2\mu})$$
$$-\frac{1}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}(\frac{\rho\rho^{lq}}{\phi^{lq}}-\rho^{lq^2})-(\frac{1}{k^{lq}}\frac{\rho^{g}}{\phi^{g}}+\frac{1}{k^{g}}\frac{\rho^{lq}}{\phi^{lq}})(\frac{N^{gg}}{c_k^2}-\frac{\rho N^{gg}}{\lambda+2\mu}+\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{c_k^2})-\frac{N^{gg}}{\lambda+2\mu}(\frac{\rho^{lq^2}}{k^{g}}+\frac{\rho^{g^2}}{k^{lq}})$$

$$a_{2k1}=\frac{2\zeta_k}{c_k^2}\frac{1}{k^{lq}}(\frac{2}{c_k^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{g}\rho^{g}}{\lambda+2\mu})-\frac{1}{k^{lq}k^{g}}(\frac{N^{gg}}{c_k^2}-\frac{\rho N^{gg}}{\lambda+2\mu}+\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{c_k^2})$$
$$-\frac{2\zeta_k}{c_k^2}(\frac{1}{k^{lq}}\frac{\rho^{g}}{\phi^{g}}+\frac{1}{k^{g}}\frac{\rho^{lq}}{\phi^{lq}})(N^{gg}+\frac{b^{g^2}}{\lambda+2\mu})+\frac{4\zeta_k^2}{c_k^4}\frac{\rho^{lq}}{\phi^{lq}}$$

$$a_{2k0}=\frac{2\zeta_k}{c_k^2}(\frac{2\zeta_k}{c_k^2}\frac{1}{k^{lq}}-\frac{N^{gg}}{k^{lq}k^{g}}-\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{k^{lq}k^{g}})$$

$$a_{233}=[\frac{1}{c_3^2}\frac{\rho^{lq}}{\phi^{lq}}(\frac{1}{c_3^2}-\frac{\rho}{\lambda+2\mu}+\frac{2b^{g}\rho^{g}}{\lambda+2\mu})+\frac{\rho^{lq^2}}{\lambda+2\mu}\frac{1}{c_3^2}-\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^{g}}{\phi^{g}}(\frac{N^{gg}}{c_3^2}-\frac{\rho N^{gg}}{\lambda+2\mu}+\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{c_3^2})-\frac{\rho^{lq}\rho^{g}N^{gg}}{\lambda+2\mu}(\frac{\rho^{lq}}{\phi^{g}}+\frac{\rho^{g}}{\phi^{lq}})]$$

$$a_{312}=\frac{1}{2\pi}[\frac{N^{lqg}}{V_P^2}\frac{\rho^{lq}\rho^{g}}{\phi^{lq}\phi^{g}}-\frac{N^{lqg}\rho^{lq}\rho^{g}}{\lambda+2\mu}(\frac{\rho}{\phi^{lq}\phi^{g}}-\frac{\rho^{lq}}{\phi^{g}}-\frac{\rho^{g}}{\phi^{lq}})+\frac{\rho^{lq}\rho^{g}}{\lambda+2\mu}\frac{1}{V_P^2}(\frac{b^{lq}b^{g}}{\phi^{lq}\phi^{g}}-\frac{b^{lq}}{\phi^{lq}}-\frac{b^{g}}{\phi^{g}}+1)]$$

$$a_{311} = \frac{1}{2\pi}[\frac{N^{lqg}}{V_P^2}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) - \frac{N^{lqg}}{\lambda+2\mu}(\rho(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) - \rho^{lq^2}\frac{1}{k^g} - \rho^{g^2}\frac{1}{k^{lq}})$$
$$+\frac{1}{\lambda+2\mu}\frac{1}{V_P^2}(b^{lq}b^g(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) - b^{lq}\rho^g\frac{1}{k^{lq}} - b^g\rho^{lq}\frac{1}{k^g})]$$

$$a_{310} = \frac{1}{2\pi}\frac{1}{k^{lq}k^g}[\frac{N^{lqg}}{V_P^2} + \frac{b^{lq}b^g}{\lambda+2\mu}\frac{1}{V_P^2} - \frac{\rho N^{lqg}}{\lambda+2\mu}]$$

$$a_{3k3} = \frac{N^{lqg}}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g} - \frac{N^{lqg}\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{\rho}{\phi^{lq}\phi^g} - \frac{\rho^{lq}}{\phi^g} - \frac{\rho^g}{\phi^{lq}}) + \frac{\rho^{lq}\rho^g}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^{lq}b^g}{\phi^{lq}\phi^g} - \frac{b^{lq}}{\phi^{lq}} - \frac{b^g}{\phi^g} + 1)$$

$$a_{3k2} = \frac{N^{lqg}}{c_k^2}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) + N^{lqg}\frac{2\zeta_k}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g} + \frac{b^{lq}b^g}{\lambda+2\mu}\frac{1}{c_k^2}[\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}} - \frac{\rho^g}{k^{lq}b^g} - \frac{\rho^{lq}}{k^g b^{lq}}]$$
$$+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}(\frac{b^{lq}b^g}{\phi^{lq}\phi^g} - \frac{b^{lq}}{\phi^{lq}} - \frac{b^g}{\phi^g} + 1) - \frac{N^{lqg}}{\lambda+2\mu}[\rho(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) - \frac{\rho^{lq^2}\rho^g}{\phi^g} - \frac{\rho^{g^2}\rho^{lq}}{\phi^{lq}}]$$

$$a_{3k1} = \frac{1}{k^{lq}k^g c_k^2}(N^{lqg} - \frac{\rho N^{lqg}c_k^2}{\lambda+2\mu} + \frac{b^{lq}b^g}{\lambda+2\mu}) + N^{lqg}\frac{2\zeta_k}{c_k^2}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) + \frac{b^{lq}b^g}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}[\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}} - \frac{\rho^g}{k^{lq}b^g} - \frac{\rho^{lq}}{k^g b^{lq}}]$$

$$a_{3k0} = \frac{2\zeta_k}{c_k^2}\frac{1}{k^{lq}k^g}[N^{lqg} + \frac{b^{lq}b^g}{\lambda+2\mu}]$$

$$a_{413} = \frac{x_i}{r}\frac{1}{V_P}[\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}(\frac{b^g N^{lqlq} - b^{lq}N^{lqg}}{\lambda+2\mu}) - \frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{\phi^{lq}} - \frac{N^{lqg}}{\phi^g}) + \frac{\rho^g}{\lambda+2\mu}\frac{1}{V_P^2}(1 - \frac{b^g}{\phi^g})]$$

$$a_{412} = \frac{x_i}{r}\frac{1}{V_P}[(\frac{b^g N^{lqlq} - b^{lq}N^{lqg}}{\lambda+2\mu})(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) - \frac{1}{\lambda+2\mu}(\frac{\rho^g N^{lqlq}}{k^{lq}} - \frac{\rho^{lq}N^{lqg}}{k^g} + \frac{b^g}{k^g}\frac{1}{V_P^2})]$$

$$a_{411} = \frac{x_i}{r}\frac{1}{V_P}\frac{1}{k^{lq}k^g}[\frac{b^g N^{lqlq} - b^{lq}N^{lqg}}{\lambda+2\mu}]$$

$$a_{4k3} = \frac{x_i}{r}\frac{1}{c_k^2}[\frac{\rho^g}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^g}{\phi^g} - 1) + (\frac{b^{lq}N^{lqg} - b^g N^{lqlq}}{\lambda+2\mu})\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g} + \frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{\phi^{lq}} - \frac{N^{lqg}}{\phi^g})]$$

$$a_{4k2} = \frac{x_i}{r}\{\frac{1}{c_k^2}[\frac{b^g}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^g} + \frac{2\zeta_k}{c_k^2}\frac{\rho^g}{\phi^g}) + \frac{b^{lq}N^{lqg} - b^g N^{lqlq}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) + \frac{1}{\lambda+2\mu}(\frac{\rho^g N^{lqlq}}{k^{lq}} - \frac{\rho^{lq}N^{lqg}}{k^g} - \rho^g\frac{2\zeta_k}{c_k^2})]$$
$$+\frac{2\zeta_k}{c_k^2}[\frac{\rho^g}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^g}{\phi^g} - 1) + \frac{b^{lq}N^{lqg} - b^g N^{lqlq}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g} + \frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{\phi^{lq}} - \frac{N^{lqg}}{\phi^g})]\}$$

$$a_{4k1} = \frac{x_i}{r}\{\frac{1}{\lambda+2\mu}\frac{1}{c_k^2}\frac{1}{k^g}(b^g\frac{2\zeta_k}{c_k^2} + \frac{b^{lq}N^{lqg} - b^g N^{lqlq}}{k^{lq}}) +$$
$$\frac{2\zeta_k}{c_k^2}[\frac{b^g}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^g} + \frac{2\zeta_k}{c_k^2}\frac{\rho^g}{\phi^g}) + \frac{b^{lq}N^{lq}g - b^g N^{lqlq}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}}) + \frac{1}{\lambda+2\mu}(\frac{\rho^g N^{lqlq}}{k^{lq}} - \frac{\rho^{lq}N^{lqg}}{k^g} - \rho^g\frac{2\zeta_k}{c_k^2})]\}$$

$$a_{4k0} = \frac{x_i}{r}\frac{2\zeta_k}{c_k^2}\frac{1}{\lambda+2\mu}\frac{1}{k^g}[b^g\frac{2\zeta_k}{c_k^2} + \frac{b^{lq}N^{lqg} - b^g N^{lqlq}}{k^{lq}}]$$

$$a_{513} = \frac{x_i}{r}\frac{1}{V_P}[\frac{b^{lq}N^{gg} - b^g N^{lqg}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g} + \frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqg}}{\phi^{lq}} - \frac{N^{gg}}{\phi^g}) + \frac{\rho^{lq}}{\lambda+2\mu}\frac{1}{V_P^2}(1 - \frac{b^{lq}}{\phi^{lq}})]$$

$$a_{512}=\frac{x_i}{r}\frac{1}{V_P}[\frac{b^{lq}N^{gg}-b^gN^{lqg}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{1}{\lambda+2\mu}(\frac{\rho^gN^{lqg}}{k^{lq}}-\frac{\rho^{lq}N^{gg}}{k^g}-\frac{b^{lq}}{k^{lq}}\frac{1}{V_P^2})]$$

$$a_{511}=\frac{x_i}{r}\frac{1}{V_P}\frac{1}{k^{lq}}\frac{1}{k^g}[\frac{b^{lq}N^{gg}-b^gN^{lqg}}{\lambda+2\mu}]$$

$$a_{5k3}=\frac{x_i}{r}\frac{1}{c_k^2}[\frac{\rho^{lq}}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^{lq}}{\phi^{lq}}-1)+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{gg}}{\phi^g}-\frac{N^{lqg}}{\phi^{lq}})]$$

$$\begin{aligned}a_{5k2}=\frac{x_i}{r}\{&\frac{1}{c_k^2}[\frac{b^{lq}}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^{lq}}+\frac{2\zeta_k}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}})+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{1}{\lambda+2\mu}(\frac{\rho^{lq}N^{gg}}{k^g}-\frac{\rho^gN^{lqg}}{k^{lq}}-\rho^{lq}\frac{2\zeta_k}{c_k^2})]\\&+\frac{2\zeta_k}{c_k^2}[\frac{\rho^{lq}}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^{lq}}{\phi^{lq}}-1)+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{gg}}{\phi^g}-\frac{N^{lqg}}{\phi^{lq}})]\}\end{aligned}$$

$$\begin{aligned}a_{5k1}=\frac{x_i}{r}\{&\frac{1}{c_k^2}(\frac{b^{lq}}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}\frac{1}{k^{lq}}+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}\frac{1}{k^{lq}}\frac{1}{k^g})+\\&\frac{2\zeta_k}{c_k^2}[\frac{b^{lq}}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^{lq}}+\frac{2\zeta_k}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}})+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{1}{\lambda+2\mu}(\frac{\rho^{lq}N^{gg}}{k^g}-\frac{\rho^gN^{lqg}}{k^{lq}}-\rho^{lq}\frac{2\zeta_k}{c_k^2})]\}\end{aligned}$$

$$a_{5k0}=\frac{x_i}{r}\frac{2\zeta_k}{c_k^2}\frac{1}{\lambda+2\mu}\frac{1}{k^{lq}}[b^{lq}\frac{2\zeta_k}{c_k^2}+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{k^g}]$$

$$a_{612}=\frac{x_i}{r}\frac{1}{V_P}[\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}(\frac{b^gN^{lqlq}-b^{lq}N^{lqg}}{\lambda+2\mu})-\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{\phi^{lq}}-\frac{N^{lqg}}{\phi^g})+\frac{\rho^g}{\lambda+2\mu}\frac{1}{V_P^2}(1-\frac{b^g}{\phi^g})]$$

$$a_{611}=\frac{x_i}{r}\frac{1}{V_P}[(\frac{b^gN^{lqlq}-b^{lq}N^{lqg}}{\lambda+2\mu})(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})-\frac{1}{\lambda+2\mu}(\frac{\rho^gN^{lqlq}}{k^{lq}}-\frac{\rho^{lq}N^{lqg}}{k^g}+\frac{b^g}{k^g}\frac{1}{V_P^2})]$$

$$a_{610}=\frac{x_i}{r}\frac{1}{V_P}\frac{1}{k^{lq}k^g}[\frac{b^gN^{lqlq}-b^{lq}N^{lqg}}{\lambda+2\mu}]$$

$$a_{6k3}=\frac{x_i}{r}\frac{1}{c_k^2}[\frac{\rho^g}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^g}{\phi^g}-1)+(\frac{b^{lq}N^{lqg}-b^gN^{lqlq}}{\lambda+2\mu})\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{\phi^{lq}}-\frac{N^{lqg}}{\phi^g})]$$

$$\begin{aligned}a_{6k2}=\frac{x_i}{r}\{&\frac{1}{c_k^2}[\frac{b^g}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^g}+\frac{2\zeta_k}{c_k^2}\frac{\rho^g}{\phi^g})+\frac{b^{lq}N^{lqg}-b^gN^{lqlq}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{k^{lq}}-\frac{\rho^{lq}N^{lqg}}{\rho^gk^g}-\frac{2\zeta_k}{c_k^2})]\\&+\frac{2\zeta_k}{c_k^2}[\frac{\rho^g}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^g}{\phi^g}-1)+\frac{b^{lq}N^{lqg}-b^gN^{lqlq}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{\phi^{lq}}-\frac{N^{lqg}}{\phi^g})]\}\end{aligned}$$

$$\begin{aligned}a_{6k1}=\frac{x_i}{r}\{&\frac{1}{\lambda+2\mu}\frac{1}{c_k^2}\frac{1}{k^g}(b^g\frac{2\zeta_k}{c_k^2}+\frac{b^{lq}N^{lqg}-b^gN^{lqlq}}{k^{lq}})+\\&\frac{2\zeta_k}{c_k^2}[\frac{b^g}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^g}+\frac{2\zeta_k}{c_k^2}\frac{\rho^g}{\phi^g})+\frac{b^{lq}N^{lq}g-b^gN^{lqlq}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{\rho^g}{\lambda+2\mu}(\frac{N^{lqlq}}{k^{lq}}-\frac{\rho^{lq}N^{lqg}}{\rho^gk^g}-\frac{2\zeta_k}{c_k^2})]\}\end{aligned}$$

$$a_{6k0}=\frac{x_i}{r}\frac{2\zeta_k}{c_k^2}\frac{1}{\lambda+2\mu}\frac{1}{k^g}[b^g\frac{2\zeta_k}{c_k^2}+\frac{b^{lq}N^{lqg}-b^gN^{lqlq}}{k^{lq}}]$$

$$a_{712}=\frac{x_i}{r}\frac{1}{V_P}[\frac{b^{lq}N^{gg}-b^gN^{lqg}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{lqg}}{\phi^{lq}}-\frac{N^{gg}}{\phi^g})+\frac{\rho^{lq}}{\lambda+2\mu}\frac{1}{V_P^2}(1-\frac{b^{lq}}{\phi^{lq}})]$$

$$a_{711}=\frac{x_i}{r}\frac{1}{V_P}[\frac{b^{lq}N^{gg}-b^gN^{lqg}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{1}{\lambda+2\mu}(\frac{\rho^gN^{lqg}}{k^{lq}}-\frac{\rho^{lq}N^{gg}}{k^g}-\frac{b^{lq}}{k^{lq}}\frac{1}{V_P^2})]$$

$$a_{710}=\frac{x_i}{r}\frac{1}{V_P}\frac{1}{k^{lq}}\frac{1}{k^g}[\frac{b^{lq}N^{gg}-b^gN^{lqg}}{\lambda+2\mu}]$$

$$a_{7k3}=\frac{x_i}{r}\frac{1}{c_k^2}[\frac{\rho^{lq}}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^{lq}}{\phi^{lq}}-1)+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{gg}}{\phi^g}-\frac{N^{lqg}}{\phi^{lq}})]$$

$$a_{7k2}=\frac{x_i}{r}\{\frac{1}{c_k^2}[\frac{b^{lq}}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^{lq}}+\frac{2\zeta_k}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}})+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{\rho^{lq}}{\lambda+2\mu}(\frac{N^{gg}}{k^g}-\frac{\rho^gN^{lqg}}{\rho^{lq}k^{lq}}-\frac{2\zeta_k}{c_k^2})]$$
$$+\frac{2\zeta_k}{c_k^2}[\frac{\rho^{lq}}{\lambda+2\mu}\frac{1}{c_k^2}(\frac{b^{lq}}{\phi^{lq}}-1)+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}+\frac{\rho^{lq}\rho^g}{\lambda+2\mu}(\frac{N^{gg}}{\phi^g}-\frac{N^{lqg}}{\phi^{lq}})]\}$$

$$a_{7k1}=\frac{x_i}{r}\{\frac{1}{c_k^2}(\frac{b^{lq}}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}\frac{1}{k^{lq}}+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}\frac{1}{k^{lq}}\frac{1}{k^g})+$$
$$\frac{2\zeta_k}{c_k^2}[\frac{b^{lq}}{\lambda+2\mu}(\frac{1}{c_k^2}\frac{1}{k^{lq}}+\frac{2\zeta_k}{c_k^2}\frac{\rho^{lq}}{\phi^{lq}})+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{\lambda+2\mu}(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})+\frac{\rho^{lq}}{\lambda+2\mu}(\frac{N^{gg}}{k^g}-\frac{\rho^gN^{lqg}}{\rho^{lq}k^{lq}}-\frac{2\zeta_k}{c_k^2})]\}$$

$$a_{7k0}=\frac{x_i}{r}\frac{2\zeta_k}{c_k^2}\frac{1}{\lambda+2\mu}\frac{1}{k^{lq}}[b^{lq}\frac{2\zeta_k}{c_k^2}+\frac{b^gN^{lqg}-b^{lq}N^{gg}}{k^g}]$$

$$a_{840}=\frac{1}{k^{lq}k^g}(\rho\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}-\rho^{lq^2}\frac{\rho^g}{\phi^g}-\rho^{g^2}\frac{\rho^{lq}}{\phi^{lq}})^{-1}$$

$$a_{841}=(\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g}+\frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}})(\rho\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}-\rho^{lq^2}\frac{\rho^g}{\phi^g}-\rho^{g^2}\frac{\rho^{lq}}{\phi^{lq}})^{-1}$$

$$a_{842}=\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}(\rho\frac{\rho^{lq}}{\phi^{lq}}\frac{\rho^g}{\phi^g}-\rho^{lq^2}\frac{\rho^g}{\phi^g}-\rho^{g^2}\frac{\rho^{lq}}{\phi^{lq}})^{-1}$$

$$a_{8k2}=\frac{1}{\lambda+2\mu}\frac{1}{c_k^2}[\frac{\rho}{\lambda+2\mu}-(\frac{1}{V_P^2}+\frac{1}{c_j^2}-\frac{b^{lq^2}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}+\frac{2\rho^{lq}b^{lq}}{\lambda+2\mu}-\frac{b^{g^2}}{\lambda+2\mu}\frac{\rho^g}{\phi^g}+\frac{2\rho^gb^g}{\lambda+2\mu})]\quad j,k=2,3\quad j\neq k$$

$$a_{8k1}=\frac{1}{\lambda+2\mu}[\frac{\rho}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}-\frac{1}{c_k^2}(\frac{2\zeta_j}{c_j^2}-\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{k^{lq}}-\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{k^g})-$$
$$\frac{2\zeta_k}{c_k^2}(\frac{1}{V_P^2}+\frac{1}{c_j^2}-\frac{b^{lq^2}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}}+\frac{2\rho^{lq}b^{lq}}{\lambda+2\mu}-\frac{b^{g^2}}{\lambda+2\mu}\frac{\rho^g}{\phi^g}+\frac{2\rho^gb^g}{\lambda+2\mu})]\qquad j,k=2,3\quad j\neq k$$

$$a_{8k0}=-\frac{1}{\lambda+2\mu}\frac{2\zeta_k}{c_k^2}(\frac{2\zeta_j}{c_j^2}-\frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{k^{lq}}-\frac{b^{g^2}}{\lambda+2\mu}\frac{1}{k^g})\qquad j,k=2,3\quad j\neq k$$

$$a_{934}=V_P^{-2}c_2^{-2}c_3^{-2}a_{842}$$

$$a_{933}=V_P^{-2}c_2^{-2}c_3^{-2}a_{841}+V_P^{-2}(2\zeta_3c_2^{-2}c_3^{-2}+2\zeta_2c_2^{-2}c_3^{-2})a_{842}$$

$$a_{932}=V_P^{-2}c_2^{-2}c_3^{-2}a_{840}+V_P^{-2}(2\zeta_3c_2^{-2}c_3^{-2}+2\zeta_2c_2^{-2}c_3^{-2})a_{841}+2\zeta_2 2\zeta_3V_P^{-2}c_2^{-2}c_3^{-2}a_{842}$$

$$a_{931}=V_P^{-2}(2\zeta_3c_2^{-2}c_3^{-2}+2\zeta_2c_2^{-2}c_3^{-2})a_{840}+2\zeta_2 2\zeta_3V_P^{-2}c_2^{-2}c_3^{-2}a_{841}$$

$$a_{930} = 2\zeta_2 2\zeta_3 V_P^{-2} c_2^{-2} c_3^{-2} a_{840}$$

$$a_{9nm} = a_{93m} \quad n = 1,2 \quad m = 0,1,2,3,4$$

$$a_{811} = \frac{1}{\lambda+2\mu}\frac{1}{V_P^2}[\frac{\rho}{\lambda+2\mu} - (\frac{1}{c_2^2} + \frac{1}{c_3^2} - \frac{b^{lq^2}}{\lambda+2\mu}\frac{\rho^{lq}}{\phi^{lq}} + \frac{2\rho^{lq}b^{lq}}{\lambda+2\mu} - \frac{b^{g^2}}{\lambda+2\mu}\frac{\rho^g}{\phi^g} + \frac{2\rho^g b^g}{\lambda+2\mu})]$$

$$a_{810} = -\frac{1}{\lambda+2\mu}\frac{1}{V_P^2}(\frac{2\zeta_2}{c_2^2} + \frac{2\zeta_3}{c_3^2} - \frac{b^{lq^2}}{\lambda+2\mu}\frac{1}{k^{lq}} - \frac{b^{g^2}}{\lambda+2\mu}\frac{1}{k^g})$$

$$b_{111} = c_2^{-2} - V_P^{-2}, \quad b_{112} = 2\zeta_2 c_2^{-2}, \quad b_{113} = c_3^{-2} - V_P^{-2}, \quad b_{114} = 2\zeta_3 c_3^{-2}$$

$$b_{120} = -2\pi, \quad b_{121} = c_2^{-2} - V_P^{-2}, \quad b_{122} = 2\zeta_2 c_2^{-2}, \quad b_{123} = c_3^{-2} - c_2^{-2}, \quad b_{124} = 2(\zeta_3 c_3^{-2} - \zeta_2 c_2^{-2})$$

$$b_{130} = 2\pi, \quad b_{131} = c_3^{-2} - V_P^{-2}, \quad b_{132} = 2\zeta_3 c_3^{-2}, \quad b_{133} = c_3^{-2} - c_2^{-2}, \quad b_{134} = 2(\zeta_3 c_3^{-2} - \zeta_2 c_2^{-2})$$

$$b_{320} = b_{220} = -2\pi, \quad b_{330} = b_{230} = 2\pi, \quad b_{3nm} = b_{2nm} = b_{1nm} \qquad n=1,2,3 \qquad m=1,2,3,4$$

$$b_{m10} = 2\pi, \quad b_{m20} = 2\pi, \quad b_{m30} = -2\pi \qquad m=4,5,6,7$$

$$b_{ynm} = b_{1nm} \qquad n=1,2,3 \qquad m=1,2,3,4 \qquad y = 4,5,6,7$$

$$b_{840} = \frac{1}{k^{lq}k^g}(\frac{\rho^{lq}\rho^g}{\phi^{lq}\phi^g} - \frac{\rho^{lq^2}\rho^g}{\rho\phi^g} - \frac{\rho^{g^2}\rho^{lq}}{\rho\phi^{lq}})^{-1}$$

$$b_{841} = (\frac{1}{k^{lq}}\frac{\rho^g}{\phi^g} + \frac{1}{k^g}\frac{\rho^{lq}}{\phi^{lq}} - \frac{\rho^{lq^2}}{\rho k^g} - \frac{\rho^{g^2}}{\rho k^{lq}})(\frac{\rho^{lq}\rho^g}{\phi^{lq}\phi^g} - \frac{\rho^{lq^2}\rho^g}{\rho\phi^g} - \frac{\rho^{g^2}\rho^{lq}}{\rho\phi^{lq}})^{-1}$$

$$b_{821} = b_{831} = (2\zeta_3 c_3^{-2} - 2\zeta_2 c_2^{-2}), b_{820} = b_{830} = (c_3^{-2} - c_2^{-2})$$

$$b_{8k3} = 2\zeta_k c_k^{-2}, b_{8k2} = (c_k^{-2} - V_P^{-2}) \qquad k = 2,3$$

$$b_{811} = (c_3^{-2} - V_P^{-2}), \quad b_{812} = 2\zeta_3 c_3^{-2}, \quad b_{813} = (c_2^{-2} - V_P^{-2}), \quad b_{814} = 2\zeta_2 c_2^{-2}$$

$$c_{yk1} = \frac{b_{yk2}}{b_{yk1}}, \qquad c_{yk2} = \frac{b_{yk4}}{b_{yk3}} \qquad k = 1,2,3, \quad y = 1 \text{ to } 7$$

$$c_{841} = \frac{b_{841}}{2}, c_{840}^2 = c_{841}^2 - b_{840}, \quad c_{8k0} = \frac{b_{8k2}}{b_{8k1}}, c_{8k1} = \frac{b_{8k4}}{b_{8k3}}, \quad k = 1,2,3$$